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PHY120/165 — Unit 8
Multiple integrals and vector calculus

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Contents

Learning outcomes

Be able to perform vector calculus manipulations using index notation.
Understand Einstein's summation convention, Kronecker δ symbol, Levi-Civita ε symbol.
Understand the definition of multiple integrals of multi-variable functions.
Be able to compute double and triple integrals in cartesian and polar coordinates.
Be able to apply multiple integrals to the calculation of volumes and areas.
Understand the definition of line integrals and surface integrals of vector fields.
Know and be able to apply divergence theorem and Stokes' theorem.
Be able to perform simple calculations of fluxes and line integrals of vector fields.
Understand the application of divergence theorem and Stokes' theorem to the differential and integral formulations of Maxwell's equations of electromagnetism.

References

The material of this unit is covered well in many textbooks on introductory mathematics for science students. For instance, L. Lyons, *All you ever wanted to know about mathematics but were afraid to ask*, CUP; M. Boas, *Mathematical methods in the physical sciences*, Wiley.

1 Index notation

In this section we introduce index notation, a very useful technique to perform vector calculus manipulations, and we revisit results on vector differentiation and gradient, divergence, curl.

1.1 Cartesian components and Einstein's summation convention

We start by introducing a compact notation for the cartesian components of vector $\mathbf{v}$,

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \ .$$

First we label cartesian directions by an index i taking values $1, 2, 3$, with the correspondence $1 \to x$, $2 \to y$, $3 \to z$, and denote unit vectors in these directions by $\mathbf{e}_i$, so that

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 = \sum_{i=1}^{3} v_i \mathbf{e}_i \ .$$

Next we introduce Einstein's summation convention that repeated indices (on any one side of any equation) are understood to be summed over, and thus write

$$\mathbf{v} = v_i \mathbf{e}_i \ . \tag{1.1}$$

1.2 Application to differential operators: the gradient

We can apply this notation to differential operators. The gradient

$$\nabla = \mathbf{i} \ \frac{\partial}{\partial x} + \mathbf{j} \ \frac{\partial}{\partial y} + \mathbf{k} \ \frac{\partial}{\partial z}$$

is rewritten in index notation as

$$\nabla = \mathbf{e}_i \ \frac{\partial}{\partial x_i} \ .$$

We often use the abbreviation $\partial / \partial x_i \equiv \partial_i$ so that

$$\nabla = \mathbf{e}_i \partial_i \ . \tag{1.2}$$

We will write the gradient of a scalar field $\phi(x, y, z)$ as

$$\nabla \phi = \mathbf{e}_i \partial_i \phi \ .$$

That is, the i-th component of the gradient of ϕ is

$$(\nabla \phi)_i = \partial_i \phi \ . \tag{1.3}$$

1.3 "Multiplying" vectors in index notation: the δ and ε symbols

We know two ways of "multiplying" vectors:
the scalar product (or dot product)

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z$$

and the vector product (or cross product)

$$\mathbf{u} \wedge \mathbf{v} = (u_y v_z - u_z v_y)\mathbf{i} + (u_z v_x - u_x v_z)\mathbf{j} + (u_x v_y - u_y v_x)\mathbf{k}$$

How can we express these in index notation?
Introduce the Kronecker δ_{ij} symbol, defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.4}$$

Then the dot product is

$$\mathbf{u} \cdot \mathbf{v} = \delta_{ij} u_i v_j = u_i v_i \tag{1.5}$$

Next introduce the Levi-Civita ε_{ijk} symbol, defined as

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if} \quad ijk = \text{even permutation of } 1\ 2\ 3 \\ -1 & \text{if} \quad ijk = \text{odd permutation of } 1\ 2\ 3 \\ 0 & \text{otherwise} \end{cases} . \tag{1.6}$$

Then the cross product is

$$\mathbf{u} \wedge \mathbf{v} = \varepsilon_{ijk} \mathbf{e}_i u_j v_k \tag{1.7}$$

The i-th component of the cross product, in particular, can be identified from Eq (1.7) as

$$(\mathbf{u} \wedge \mathbf{v})_i = \varepsilon_{ijk} u_j v_k \tag{1.8}$$

Note:

The Kronecker δ in Eq (1.4) is symmetric under index exchange: $\delta_{ij} = \delta_{ji}$.

The Levi-Civita ε_{ijk} in Eq (1.6) is antisymmetric, that is, it changes sign under exchange of any two indices: e.g., $\varepsilon_{ijk} = -\varepsilon_{jik}$. In particular, $\varepsilon = 0$ if any two indices are equal: e.g., $\varepsilon_{iik} = 0$. For any cyclic permutation of the indices ε is left unchanged: e.g., $\varepsilon_{ijk} = \varepsilon_{kij}$.

Note:

The δ and ε symbols provide two ways of "multiplying" any two vectors, $\mathbf{u}$ and $\mathbf{v}$, corresponding to the dot product $\delta_{ij} u_i v_j$ (Eq (1.5)) and cross product $\varepsilon_{ijk} \mathbf{e}_i u_j v_k$ (Eq (1.7)). It can be shown that these are the only two ways of combining vectors in three dimensions which preserve rotational invariance.

1.4 Divergence and curl in index notation

Using the δ symbol, the divergence of any vector field $\mathbf{F}(x, y, z)$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z$$

will be written

$$\nabla \cdot \mathbf{F} = \delta_{ij} \partial_i F_j = \partial_i F_i \ . \tag{1.9}$$

Analogously, using the ε symbol the curl of vector field $\mathbf{F}(x, y, z)$

$$\nabla \wedge \mathbf{F} = \mathbf{i} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

is

$$\nabla \wedge \mathbf{F} = \varepsilon_{ijk} \mathbf{e}_i \partial_j F_k \ . \tag{1.10}$$

In particular the i-th component of the curl is

$$(\nabla \wedge \mathbf{F})_i = \varepsilon_{ijk} \partial_j F_k \ . \tag{1.11}$$

Note:

We are using cartesian coordinates for which $\partial_i \mathbf{e}_j = 0$. Generalizations of the above formulas are required for curvilinear coordinates.

Example 1.1

Divergence of the cross product: Show that for any two vector fields $\mathbf{F}$ and $\mathbf{G}$

$$\nabla \cdot \mathbf{F} \wedge \mathbf{G} = \mathbf{G} \cdot \nabla \wedge \mathbf{F} - \mathbf{F} \cdot \nabla \wedge \mathbf{G} \ .$$

Using Eq (1.9) for divergence and Eq (1.7) for cross product we have

$$\begin{aligned} \nabla \cdot \mathbf{F} \wedge \mathbf{G} &= \partial_i (\varepsilon_{ijk} F_j G_k) \\ &= \varepsilon_{ijk} (\partial_i F_j) G_k + \varepsilon_{ijk} F_j (\partial_i G_k) \end{aligned}$$

where in the last line we have applied the product rule of differentiation. Next, using a cyclic permutation of indices in the first term and an odd permutation in the second term, we get

$$\begin{aligned} \nabla \cdot \mathbf{F} \wedge \mathbf{G} &= \varepsilon_{kij} (\partial_i F_j) G_k - \varepsilon_{jik} F_j (\partial_i G_k) \\ &= (\nabla \wedge \mathbf{F})_k G_k - F_j (\nabla \wedge \mathbf{G})_j = \mathbf{G} \cdot \nabla \wedge \mathbf{F} - \mathbf{F} \cdot \nabla \wedge \mathbf{G} \end{aligned}$$

where in the last line we have used Eq (1.11) for the curl and Eq (1.5) for the dot product.

By manipulations similar to the ones in the example above we can obtain a host of vector calculus results. For instance

Example 1.2

Divergence and curl of a scalar field times a vector field:

Show that for any scalar field ϕ and vector field $\mathbf{F}$

$$\text{i)}\ \nabla\cdot(\phi\mathbf{F}) = \mathbf{F}\cdot\nabla\phi + \phi\nabla\cdot\mathbf{F}\ \ ,\quad \text{ii)}\ \nabla\wedge(\phi\mathbf{F}) = \nabla\phi\wedge\mathbf{F} + \phi\nabla\wedge\mathbf{F}\ .$$

We have

$$\begin{aligned} \text{i)}\quad \nabla\cdot(\phi\mathbf{F}) &= \partial_i(\phi F_i) \\ &= (\partial_i\phi)F_i + \phi(\partial_i F_i) \\ &= \mathbf{F}\cdot\nabla\phi + \phi\nabla\cdot\mathbf{F} \end{aligned}$$

and

$$\begin{aligned} \text{ii)}\quad \nabla\wedge(\phi\mathbf{F}) &= \varepsilon_{ijk}\mathbf{e}_i\partial_j(\phi F_k) \\ &= \varepsilon_{ijk}\mathbf{e}_i(\partial_j\phi)F_k + \varepsilon_{ijk}\mathbf{e}_i\phi(\partial_j F_k) \\ &= \nabla\phi\wedge\mathbf{F} + \phi\nabla\wedge\mathbf{F} \end{aligned}$$

Index notation can be used as well for calculations with specific fields. For instance

Example 1.3

Let $\mathbf{F}$ be the vector field $\mathbf{F} = \mathbf{x} = (x, y, z)$, and let $\mathbf{c}$ and $\mathbf{a}$ be constant vectors. Calculate

$$\text{i)}\ \nabla\cdot\mathbf{c}\wedge\mathbf{x} \quad \text{and} \quad \text{ii)}\ \nabla\wedge(\mathbf{c}\cdot\mathbf{x}\ \mathbf{a})\ .$$

We have

$$\begin{aligned} \text{i)}\quad \nabla\cdot\mathbf{c}\wedge\mathbf{x} &= \partial_i(\varepsilon_{ijk}c_j x_k) \\ &= \varepsilon_{ijk}c_j\partial_i x_k \\ &= \varepsilon_{ijk}c_j\delta_{ik} = 0 \end{aligned} \tag{1.12}$$

where in the last line we have used that $\partial_i x_k = \delta_{ik}$, and that $\varepsilon_{ijk}\delta_{ik} = 0$ because δ_{ik} is symmetric under interchange of i and k (i.e., $\delta_{ik} = \delta_{ki}$) while ε_{ijk} is antisymmetric (i.e., $\varepsilon_{ijk} = -\varepsilon_{kji}$), and

$$\begin{aligned} \text{ii)}\quad \nabla\wedge(\mathbf{c}\cdot\mathbf{x}\ \mathbf{a}) &= \varepsilon_{ijk}\mathbf{e}_i\partial_j(c_m x_m a_k) \\ &= \varepsilon_{ijk}\mathbf{e}_i c_m\delta_{jm}a_k \\ &= \varepsilon_{ijk}\mathbf{e}_i c_j a_k = \mathbf{c}\wedge\mathbf{a} \end{aligned} \tag{1.13}$$

where in the second line we have again used that $\partial_j x_m = \delta_{jm}$ and in the last line we have applied Eq (1.7) for the cross product.

Note:

Below Eq (1.12) we have used the observation that $S_{ij} A_{ij} = 0$ if S_{ij} is symmetric and A_{ij} is antisymmetric (i.e., $S_{ij} = S_{ji}$ and $A_{ij} = -A_{ji}$). This is because

$$\begin{aligned} S_{ij} = S_{ji} \ , \ \ A_{ij} = -A_{ji} \ &\Rightarrow \ S_{ij} A_{ij} = -S_{ji} A_{ji} = -S_{ij} A_{ij} \\ &\Rightarrow \ S_{ij} A_{ij} = 0 \end{aligned} \tag{1.14}$$

where in the upper line we have first used the symmetry of S and antisymmetry of A and then used the freedom to relabel indices which are summed over in Einstein's summation convention ("dummy" indices), while in the lower line we have concluded, from $SA = -SA$, that $SA = 0$.

We will now apply index notation to discuss second-order differential operators. The first example is the divergence of a gradient, namely the Laplacian operator ∇^2,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ .$$

Using Eq (1.9) and Eq (1.2) we have

$$\begin{aligned} \nabla \cdot (\nabla \phi) &= \delta_{ij} \partial_i \partial_j \phi \\ &= \partial_i \partial_i \phi = \nabla^2 \phi \end{aligned}$$

i.e.,

$$\nabla^2 = \delta_{ij} \partial_i \partial_j = \partial_i \partial_i \ . \tag{1.15}$$

Next are two of the most important theorems on vector differentiation.

1.5 Two theorems: curl grad = 0, div curl = 0

The two theorems discussed in this section follow from the observation, made around Eq (1.14), that $S_{ij} A_{ij} = 0$ if S_{ij} is symmetric and A_{ij} is antisymmetric.

The first theorem says that for any scalar field ϕ

$$\nabla \wedge \nabla \phi = 0 \tag{1.16}$$

that is,

$$\text{curl grad } \phi = 0 \ .$$

This is because $\nabla \wedge \nabla \phi = \varepsilon_{ijk} \mathbf{e}_i \partial_j \partial_k \phi = 0$, where we have used that $\partial_j \partial_k = \partial_k \partial_j$ (i.e., $\partial_j \partial_k$ is symmetric) and $\varepsilon_{ijk} = -\varepsilon_{ikj}$ (i.e., ε is antisymmetric).

So this theorem implies that the curl of any vector field which can be written as a gradient of a scalar field ϕ is zero.

The second theorem says that for any vector field $\mathbf{F}$

$$\nabla \cdot \nabla \wedge \mathbf{F} = 0 \tag{1.17}$$

that is,

$$\text{div curl } \mathbf{F} = 0 \, .$$

This is because $\nabla \cdot \nabla \wedge \mathbf{F} = \partial_i \varepsilon_{ijk} \partial_j F_k = 0$, where we have again used that $\partial_i \partial_j$ is symmetric while ε_{ijk} is antisymmetric.

So this theorem implies that the divergence of any vector field which can be written as a curl of another vector field $\mathbf{F}$ is zero.

Example 1.4

The two homogeneous (i.e., charge- and current-independent) Maxwell equations of electromagnetism are

$$\nabla \cdot \mathbf{B} = 0 \, , \tag{1.18}$$

where $\mathbf{B}$ is the magnetic field, and

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday law)} \tag{1.19}$$

where $\mathbf{E}$ is the electric field, and the right hand side denotes minus the time derivative of the magnetic field. Show that the two homogeneous Maxwell equations are automatically satisfied if the magnetic and electric fields have the form

$$\mathbf{B} = \nabla \wedge \mathbf{A} \ \ , \quad \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{1.20}$$

for some scalar field ϕ and vector field $\mathbf{A}$.

We can reason as follows. Theorem (1.17) implies that if $\mathbf{B} = \nabla \wedge \mathbf{A}$ then $\nabla \cdot \mathbf{B} = \nabla \cdot \nabla \wedge \mathbf{A} = 0$, and so Maxwell equation (1.18) is satisfied. By substituting $\mathbf{B} = \nabla \wedge \mathbf{A}$ into Faraday law (1.19) we have

$$\nabla \wedge \mathbf{E} = -\nabla \wedge \frac{\partial \mathbf{A}}{\partial t}$$

where we have interchanged the order of time and spatial derivatives in the right hand side. That is,

$$\nabla \wedge \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \, .$$

Theorem (1.16) implies that this equation is satisfied if $(\mathbf{E} + \partial \mathbf{A}/\partial t)$ is a gradient, i.e.,

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi$$

for a scalar field ϕ, that is,

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \, .$$

Note:

The scalar ϕ and the vector $\mathbf{A}$ in Eq (1.20) are respectively the electric scalar potential and the magnetic vector potential of electromagnetism. In the case of stationary processes independent of time, we have $\partial \mathbf{A}/\partial t = 0$, and we are thus left with the relation between electric field and electric potential $\mathbf{E} = -\nabla \phi$, valid in the case of electrostatics.

1.6 A useful relation between ε and δ

Suppose we are to calculate a double cross product of three vectors **u**, **v**, **w**. What does this look like in index notation? We have

$$\begin{aligned}\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) &= \varepsilon_{ijk} \mathbf{e}_i u_j \varepsilon_{kmn} v_m w_n \\ &= \varepsilon_{kij} \varepsilon_{kmn} \mathbf{e}_i u_j v_m w_n \ .\end{aligned}$$

With a little thought one can see that the product $\varepsilon_{kij}\varepsilon_{kmn}$ can be expressed in terms of Kronecker δ's as follows,

$$\varepsilon_{kij}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \ . \tag{1.21}$$

This is because if the elements of the pair ij do not match those in the pair mn, in any order, the product of the two ε is 0, while if they do, the two possible orderings in the ε's give $+1$ or -1, as do the two terms with the δ's in the right hand side.

Then we get

$$\begin{aligned}\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) &= \delta_{im}\delta_{jn} \mathbf{e}_i u_j v_m w_n - \delta_{in}\delta_{jm} \mathbf{e}_i u_j v_m w_n \\ &= \mathbf{e}_i u_j v_i w_j - \mathbf{e}_i u_j v_j w_i \\ &= \mathbf{v}\ \mathbf{u} \cdot \mathbf{w} - \mathbf{w}\ \mathbf{u} \cdot \mathbf{v} \ ,\end{aligned}$$

which is one of the basic vector identities for the double cross product.

The identity (1.21) can be applied to computations involving differential operators as well. An example is the double curl of a vector field:

$$\begin{aligned}\nabla \wedge (\nabla \wedge \mathbf{F}) &= \varepsilon_{ijk} \mathbf{e}_i \partial_j \varepsilon_{kmn} \partial_m F_n \\ &= \mathbf{e}_i \partial_j \partial_m F_n (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \\ &= \mathbf{e}_i \partial_j \partial_i F_j - \mathbf{e}_i \partial_j \partial_j F_i \\ &= \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}\end{aligned} \tag{1.22}$$

where in the first term of the last line we have the 2nd-order "grad-div" operator

$$(\nabla(\nabla \cdot \mathbf{F}))_i = \partial_i \partial_j F_j \ \ , \tag{1.23}$$

while in the second term of the last line we have the 2nd-order "div-grad" operator (1.15), the Laplacian, applied to vector field **F** component by component,

$$(\nabla^2 \mathbf{F}))_i = \partial_j \partial_j F_i \ \ .$$

Note:

Eqs (1.15), (1.16), (1.17), (1.22), (1.23) summarize the basic results on 2nd-order differential operators of vector calculus. In particular Eqs (1.16) and (1.17) state that curl grad and div curl are zero, while Eq (1.22) gives the relation between the three non-zero operators, curl curl = grad div − div grad.

Example 1.5

In electromagnetism the identity (1.22) is essential to obtain, from Maxwell's equations, the equation which describes the propagation of electromagnetic waves. Consider a region of space free of any electromagnetic sources, i.e., free of electric charges and currents. In a source-free region Gauss law is given by

$$\nabla \cdot \mathbf{E} = 0 \tag{1.24}$$

and Ampere-Maxwell law is given by

$$\nabla \wedge \mathbf{B} = \mu_0 \varepsilon_0 \, \frac{\partial \mathbf{E}}{\partial t} \, , \tag{1.25}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, μ_0 and ε_0 are physical constants, the magnetic permeability and electric permittivity. Faraday law is

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \, . \tag{1.26}$$

By evaluating the curl of the curl of $\mathbf{E}$, show that the electric field satisfies the 2nd-order equation

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \, \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \, . \tag{1.27}$$

We can proceed as follows. Take the curl on both sides of Faraday law (1.26), and apply the identity (1.22). We thus obtain

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \, \nabla \wedge \mathbf{B} \, .$$

The first term on the left hand side is zero because of Gauss law (1.24). By evaluating the right hand side using Ampere-Maxwell law (1.25), we get

$$-\nabla^2 \mathbf{E} = -\mu_0 \varepsilon_0 \, \frac{\partial^2 \mathbf{E}}{\partial t^2} \, ,$$

that is,

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \, \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \, . \tag{1.28}$$

Eq (1.28) is the wave equation and describes the propagation of electromagnetic waves with wave speed $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

1.7 Rotational invariance of δ and ε

Rotations are described by matrices R which satisfy

$$RR^T = R^T R = I \ , \tag{1.29}$$

where R^T is the transpose matrix and I is the identity matrix, and

$$\det R = 1 \ , \tag{1.30}$$

where det is the determinant. Transformations satisfying Eq (1.29) are called orthogonal. For these, $R^T = R^{-1}$. Any vector $\mathbf{v}$ transforms under rotation R as

$$v'_i = R_{ij} v_j \ , \tag{1.31}$$

while any scalar s is left unchanged,

$$s' = s \ .$$

Here primed variables denote "rotated" quantities, $\mathbf{v}' = R\mathbf{v}$, $s' = Rs$.

The scalar product and vector product constructed via the δ and ε symbols (Eqs (1.5) and (1.7)) are the two and only two ways of "multiplying" three-dimensional vectors which preserve rotational invariance, i.e., for any two vectors $\mathbf{u}$ and $\mathbf{v}$,

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v}' \ , \tag{1.32}$$

$$(\mathbf{u} \wedge \mathbf{v})' = \mathbf{u}' \wedge \mathbf{v}' \ . \tag{1.33}$$

Example 1.6

Show Eq (1.32).

Applying the rotation (1.31) to $\mathbf{u}$ and $\mathbf{v}$, we have

$$\mathbf{u}' \cdot \mathbf{v}' = \delta_{ij} u'_i v'_j = \delta_{ij} R_{im} u_m R_{jn} v_n = R^T_{mi} \delta_{ij} R_{jn} u_m v_n \ .$$

But this equals

$$\mathbf{u} \cdot \mathbf{v} = \delta_{mn} u_m v_n$$

because due to Eq (1.29)

$$R^T_{mi} \delta_{ij} R_{jn} = \delta_{mn} \ . \tag{1.34}$$

Thus Eq (1.32) is verified.

Note:

Eq (1.33) can be verified in an analogous manner, based on the identity

$$\varepsilon_{\ell mn} = \varepsilon_{ijk} R_{i\ell} R_{jm} R_{kn} \tag{1.35}$$

which obtains under the conditions (1.29) and (1.30).

Note:

Eqs (1.34) and (1.35) express the rotational invariance of the δ and ε symbols. It can be shown that δ and ε are the only invariant symbols under three-dimensional rotations. It can also be shown that the structure of rotationally invariant symbols changes in a different number of dimensions, for instance in two and in four dimensions (which is important for the theory of space-time in relativity).

2 Multiple integrals

This section discusses how the integral of functions $f(x)$ of one real variable

$$\int_a^b f(x)\ dx \tag{2.1}$$

can be extended to functions of multiple real variables. For example, functions defined in three-dimensional space, $f(x, y, z)$, or in general functions of n real variables $f(x_1, x_2, \ldots, x_n)$.

2.1 The one-dimensional case

We begin by recalling the case of functions of a single real variable. The integral (2.1) may be introduced as follows.

i) We make a partition of the interval $[a, b]$ into N subintervals, specified by points x_i, $i = 0, 1, 2, \ldots, N$, with $a = x_0 < x_1 < x_2 < \ldots < x_N = b$.

ii) We construct the sum

$$S = \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1}) \ \ , \tag{2.2}$$

where x_i^* is a point in the subinterval (x_{i-1}, x_i).

iii) We take the limit of a finer and finer partition, i.e., smaller and smaller subintervals $|x_i - x_{i-1}| \to 0$, letting $N \to \infty$. Under suitable hypotheses on the function f the limit of the sum S exists and is finite. This defines the integral of f over $[a, b]$,

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1}) \equiv \int_a^b f(x)\ dx \ \ . \tag{2.3}$$

This construction illustrates the geometric meaning of integration. The generic term in the sum (2.2) is the area of the rectangle constructed on the subinterval (x_{i-1}, x_i) with height $f(x_i^*)$. In the limit $|x_i - x_{i-1}| \to 0$ the sum of the areas of the rectangles gives the area under the graph of the function $f(x)$ between a and b.

The above method defines the Riemann integral and can be extended to the multi-dimensional case considering functions of multiple variables. In the next subsection we describe this for the next simplest case, functions of two variables $f(x, y)$, which leads us to consider double integrals.

2.2 Double integrals

Consider a function of two real variables $f(x, y)$. Let us construct the integral of f over a region R contained in its domain of definition by generalizing the method of the previous subsection.

i) We subdivide the region R into N subregions of area ΔA_i, $i = 1, 2, \dots, N$. Let (x_i, y_i) be the coordinates of a point in the i-th subregion.

ii) We construct the sum

$$S = \sum_{i=1}^{N} f(x_i, y_i) \ \Delta A_i \quad . \tag{2.4}$$

iii) We take the limit in which the area of each subregion becomes vanishingly small, $\Delta A_i \to 0$, and $N \to \infty$. Provided this limit exists and is finite, we define the integral of $f(x, y)$ over R to be the limit of the sum S,

$$\int_R f(x, y) \ dA = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i, y_i) \ \Delta A_i \quad . \tag{2.5}$$

Taking subregions to be rectangles of area $\Delta A = \Delta x \Delta y$ we can also write the integral (2.5) as the "double integral"

$$\int_R f(x, y) \ dA = \int_R f(x, y) \ dx \ dy \quad . \tag{2.6}$$

The double integral has the geometric meaning of volume under the surface $z = f(x, y)$. This provides the analogue of the interpretation in terms of area for the single-variable integral in the previous subsection.

We may be more explicit and write the double integral, by specifying the order of integration, as "iterated", or "nested", integrals, as follows

$$\begin{aligned} \int_R f(x, y) \ dA &= \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \ f(x, y) \equiv \int_{x_1}^{x_2} \left\{ \int_{y_1(x)}^{y_2(x)} f(x, y) \ dy \right\} dx \\ &= \int_{y_1}^{y_2} dy \int_{x_1(y)}^{x_2(y)} dx \ f(x, y) \equiv \int_{y_1}^{y_2} \left\{ \int_{x_1(y)}^{x_2(y)} f(x, y) \ dx \right\} dy \quad , \end{aligned} \tag{2.7}$$

where in the first line the integration in dy is understood to be done first and to be followed by the integration in dx, while in the second line the order of integration is reversed.

Example 2.1

Calculate the integral of the function $f(x, y) = 2x + y$ over the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$:

$$I = \int_R (2x + y) \ dx \ dy \quad .$$

Method 1: calculate I by applying the definition (2.5). To do this, we take a partition of the rectangle R specified by points $(x_i, y_j) = (i/n, j/n)$, with $i =$

$1, \ldots, n$ and $j = 1, \ldots, 2n$, so that $1/n^2$ is the area of each subrectangle, and we construct the sum (2.4). We have

$$S = \sum_{i=1}^{n} \sum_{j=1}^{2n} (2x_i + y_j) \frac{1}{n^2} = \sum_{i=1}^{n} \sum_{j=1}^{2n} (2 \frac{i}{n} + \frac{j}{n}) \frac{1}{n^2}$$
$$= \frac{1}{n^3} \left(2(2n) \sum_{i=1}^{n} i + n \sum_{j=1}^{2n} j \right) = \frac{1}{n^2} \left(4 \frac{n(n+1)}{2} + \frac{2n(2n+1)}{2} \right)$$
$$= \frac{1}{n^2} (4n^2 + 3n) = 4 + \frac{3}{n}$$

We evaluate the integral I by taking the $n \to \infty$ limit of the sum S:

$$I = \int_R (2x + y) \; dx \; dy = \lim_{n \to \infty} S = 4 \quad .$$

Method 2: calculate I by iterated integrals (2.7). This gives

$$I = \int_R (2x + y) \; dx \; dy$$
$$= \int_0^1 dx \int_0^2 dy \; (2x + y) = \int_0^1 dx \left[2x(y)_0^2 + \left(\frac{y^2}{2} \right)_0^2 \right]$$
$$= \int_0^1 dx (4x + 2) = (2x^2 + 2x)_0^1 = 4$$

From this example we see that method 2 is more practical. Analogously to the one-dimensional case, in most applications we do not directly apply the definition in order to evaluate double integrals. Rather, we use iterated integrals, which allows us to reduce the problem of a multiple integral to the problem of many (nested) single integrals.

We next see examples of evaluating double integrals involving non-rectangular regions of integration.

Example 2.2

Evaluate the integral of the function $f(x, y) = 3(x^2 + y^2)$ over the region $1 \leq x \leq 3$, $0 \leq y \leq x$.

$$I = \int_R 3 \; (x^2 + y^2) \; dx \; dy$$
$$= \int_1^3 dx \int_0^x dy \; 3 \; (x^2 + y^2) = \int_1^3 dx \left[3x^2 (y)_0^x + (y^3)_0^x \right]$$
$$= \int_1^3 dx \; 4x^3 = (x^4)_1^3 = 80$$

Example 2.3

Evaluate the integral of the function $f(x, y) = xy$ over the region $x \geq 0$, $y \leq 1$, $y \geq x^2$.

$$I = \int_0^1 dx \int_{x^2}^1 dy\ x\ y = \int_0^1 dx\ x \left(\frac{y^2}{2}\right)_{x^2}^1$$
$$= \int_0^1 dx\ x \left(\frac{1}{2} - \frac{x^4}{2}\right) = \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^6}{6}\right)_0^1 = \frac{1}{6}$$

Alternatively, reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} dx\ x\ y = \int_0^1 dy\ y \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} = \int_0^1 dy\ \frac{y^2}{2} = \left(\frac{y^3}{6}\right)_0^1 = \frac{1}{6}$$

Note:

In the examples considered the integration regions R and functions f are sufficiently well behaved that the interchange of the order of integration is allowed and simply results into two equivalent ways of evaluating the given integral. We do not discuss here cases for which this does not apply.

The analysis of double integrals given above can be generalized to multi-dimensional integrals. We consider the case of triple integrals next.

2.3 Volume integrals

Consider a function of three real variables $f(x, y, z)$. The integral of f over a volume V in three-dimensional space can be introduced, analogously to what is done for the two-dimensional case in the previous subsection, by i) making a partition of V into subregions of volume ΔV_i, with $i = 1, \ldots, N$, ii) constructing the Riemann sum

$$S = \sum_{i=1}^{N} f(x_i, y_i, z_i)\ \Delta V_i \quad , \tag{2.8}$$

iii) taking the limit $\Delta V_i \to 0$, or $N \to \infty$,

$$\int_V f(x, y, z)\ dV = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i, z_i)\ \Delta V_i \quad . \tag{2.9}$$

Eq (2.9) defines the volume integral of f over V. We can also notate this explicitly as a triple integral

$$\int_V f(x, y, z)\ dV = \int_V f(x, y, z)\ dx\ dy\ dz \equiv \int_V f(\mathbf{x})\ d^3\mathbf{x} \quad . \tag{2.10}$$

For functions of n variables, we generalize this to the n-dimensional multiple integral over region R

$$\int_R f(x_1, \ldots, x_n)\ dx_1\ \ldots\ dx_n = \int_R f(\mathbf{x})\ d^n\mathbf{x}\ . \tag{2.11}$$

Example 2.4

Calculate the integral of the function $f(x, y, z) = xyz$ over the three-dimensional region V defined by $x \geq 0$, $y \geq 0$, $z \geq 0$, $x^2 + y^2 + z^2 \leq 4$.

$$\begin{aligned}
I &= \int_V xyz\ dV \\
&= \int_0^2 dx \int_0^{\sqrt{4-x^2}} dy \int_0^{\sqrt{4-x^2-y^2}} dz\ x\ y\ z = \int_0^2 dx \int_0^{\sqrt{4-x^2}} dy\ x\ y\ \frac{1}{2}(4 - x^2 - y^2) \\
&= \frac{1}{2}\int_0^2 dx \left[(4x - x^3)\frac{1}{2}(4 - x^2) - x\frac{1}{4}(4 - x^2)^2\right] = \int_0^2 dx \left(2x - x^3 + \frac{x^5}{8}\right) = \frac{4}{3}
\end{aligned}$$

2.4 Change of integration variables in double integrals

For integrals over one variable

$$\int_a^b f(x)\ dx$$

we can change integration variable from x to a new variable u, defined via

$$x = x(u)\ ,$$

by

$$\int_a^b f(x)\ dx = \int_{u_a}^{u_b} f(x(u))\ \frac{dx}{du}\ du\ , \tag{2.12}$$

where $a = x(u_a)$, $b = x(u_b)$. In this subsection we discuss the generalization of this to the case in which we perform a change of integration variables in double integrals.

Consider the double integral of function $f(x, y)$ over region R

$$I = \int_R f(x, y)\ dA\ . \tag{2.13}$$

Consider the transformation of coordinates from x, y to new variables u, v,

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v)\ . \end{aligned} \tag{2.14}$$

The area element dA in the new variables u, v may be obtained by considering, for each point in the integration region, the families of level curves $u =$ constant and $v =$ constant, and taking small variations du and dv. By evaluating the area of the parallelogram constructed at the given point from the du and dv variations, we get

$$dA = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| \; du \; dv \quad . \tag{2.15}$$

That is, by introducing the matrix of first partial derivatives

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \equiv \frac{\partial(x, y)}{\partial(u, v)} \quad , \tag{2.16}$$

we find that the area element is proportional to the determinant of this matrix:

$$dA = |J| \; du \; dv \quad , \tag{2.17}$$

where

$$J = \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \quad . \tag{2.18}$$

The matrix (2.16) is termed the jacobian matrix of the transformation (2.14) and the determinant (2.18) is the jacobian determinant. So the double integral (2.13) can be recast in terms of the new coordinates u, v as

$$I = \int_{R'} f(x(u, v), y(u, v)) \; |J| \; du \; dv \quad , \tag{2.19}$$

where R' denotes the integration region expressed in terms of u, v. Eq (2.19) provides the generalization of the result (2.12) to the case of two-dimensional integration. The treatment of the n-dimensional case proceeds along similar lines and the result is given in the next subsection.

Example 2.5

Plane polar coordinates. The transformation from cartesian coordinates x, y to plane polar coordinates r, φ is defined by

$$x = r \cos \varphi$$
$$y = r \sin \varphi$$

with $0 \le r < \infty$, $0 \le \varphi \le 2\pi$. The jacobian determinant of the transformation is

$$J = \det \begin{pmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{pmatrix} = r \quad .$$

So the area element in plane polars is

$$dA = r\ dr\ d\varphi\ .$$

Example 2.6
Calculate the area of the annular region in the xy plane $x \geq 0$, $y \geq 0$, $1 \leq x^2+y^2 \leq 4$. Using polar coordinates we have

$$A = \int_1^2 r\ dr \int_0^{\pi/2} d\varphi = \left(\frac{r^2}{2}\right)_1^2 \frac{\pi}{2} = \frac{3\pi}{4}\ .$$

Example 2.7
Gaussian integral. The integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

can be evaluated using the following trick: i) consider the function of two variables $f(x,y) = e^{-x^2-y^2}$ and integrate this over the whole xy plane; ii) evaluate this double integral in cartesian coordinates and polar coordinates, and equate the results. We obtain

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = I^2$$

$$= \int_0^{\infty} r\ dr\ e^{-r^2} \int_0^{2\pi} d\varphi = \frac{1}{2}\ 2\pi = \pi$$

$$\Rightarrow\ I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\ . \quad (2.20)$$

2.5 Change of integration variables in multiple integrals

The discussion in Section 2.4 for double integrals can be extended to the change of integration variables for multiple integrals in arbitrary number of dimensions,

$$I = \int_R f(x_1, \ldots, x_n)\ dx_1\ \ldots\ dx_n\ . \quad (2.21)$$

Going from variables $x_1, \ldots, x_n$ to new variables $u_1, \ldots, u_n$ via the transformations

$$\begin{aligned} x_1 &= x_1(u_1, \ldots, u_n) \\ x_2 &= x_2(u_1, \ldots, u_n) \\ &\ldots \\ x_n &= x_n(u_1, \ldots, u_n) \end{aligned} \quad (2.22)$$

the multiple integral (2.21) is recast as

$$I = \int_{R_u} f(x_1(u_1, \ldots, u_n), \ldots, x_n(u_1, \ldots, u_n)) \; |J| \; du_1 \; \ldots \; du_n \quad , \tag{2.23}$$

where R_u denotes the n-dimensional integration region expressed in terms of the new variables u, and J is the jacobian determinant for the transformation (2.22),

$$J = \det \begin{pmatrix} \dfrac{\partial x_1}{\partial u_1} & \cdots & \dfrac{\partial x_1}{\partial u_n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_n}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_n} \end{pmatrix} \quad . \tag{2.24}$$

The volume element in n dimensions is

$$dV = dx_1 \; \ldots \; dx_n = |J| \; du_1 \; \ldots \; du_n \quad .$$

Example 2.8

Spherical polar coordinates in three dimensions. The transformation from cartesian coordinates x, y, z to spherical polar coordinates r, θ, φ is defined by

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

with $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. The jacobian determinant of the transformation is

$$J = \det \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2 \sin\theta \quad .$$

So the volume element in spherical polars is

$$dV = r^2 \; \sin\theta \; dr \; d\theta \; d\varphi \quad .$$

Cylindrical polar coordinates in three dimensions. The transformation from cartesian coordinates x, y, z to cylindrical polar coordinates r, φ, z is defined by

$$x = r \cos\varphi$$
$$y = r \sin\varphi$$
$$z = z$$

with $0 \le r < \infty$, $0 \le \varphi \le 2\pi$, $-\infty < z < \infty$. The jacobian determinant of the transformation is

$$J = \det \begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = r \quad .$$

So the volume element in cylindrical polars is

$$dV = r\ dr\ d\varphi\ dz \ \ .$$

Example 2.9

Calculate the integral in the example at the end of Section 2.3 using spherical polar coordinates:

$$I = \int_V xyz\ dV$$

where V is the region $x \geq 0$, $y \geq 0$, $z \geq 0$, $x^2 + y^2 + z^2 \leq 4$.

$$\begin{aligned}
I &= \int_V xyz\ dV \\
&= \int_0^2 dr \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi\ r^2\ \sin\theta\ r^3\ \sin^2\theta\ \sin\varphi\ \cos\varphi\ \cos\theta \\
&= \int_0^2 dr\ r^5 \int_0^{\pi/2} d\theta\ \sin^3\theta\ \cos\theta \int_0^{\pi/2} d\varphi\ \sin\varphi\ \cos\varphi \\
&= \left(\frac{r^6}{6}\right)_0^2 \left(\frac{\sin^4\theta}{4}\right)_0^{\pi/2} \left(\frac{\sin^2\varphi}{2}\right)_0^{\pi/2} = \frac{64}{6}\frac{1}{4}\frac{1}{2} = \frac{4}{3}
\end{aligned}$$

2.6 An application to mass distributions

For a distribution of mass with density $\rho = \rho(\mathbf{x})$ in region V of three-dimensional space, volume integrals serve to calculate the total mass of the distribution

$$M = \int_V \rho(\mathbf{x})\ dV \ \ , \tag{2.25}$$

the coordinates of its centre of mass

$$\mathbf{x}_{\mathrm{CM}} = \frac{1}{M} \int_V \mathbf{x}\ \rho(\mathbf{x})\ dV \ \ , \tag{2.26}$$

the moment of inertia about a given axis

$$I = \int_V d^2(\mathbf{x})\ \rho(\mathbf{x})\ dV \ \ , \tag{2.27}$$

where $d^2(\mathbf{x})$ is the squared distance of point $\mathbf{x}$ from the axis.

Analogous expressions obtain for two-dimensional mass distributions in terms of the surface mass density $\sigma(x, y)$ (instead of the volume mass density $\rho(x, y, z)$) and corresponding double integrals (instead of volume integrals).

Example 2.10

Calculate the moment of inertia about a diameter of a uniform sphere of radius R and mass M.

We may take the z axis along the diameter about which we compute the moment of inertia. For each point (x, y, z) the distance square from the axis is $x^2 + y^2$. Since the density is uniform, the mass is

$$M = \int_V \rho(\mathbf{x})\ dV = \rho\ \frac{4\pi R^3}{3}\quad , \quad \text{so} \quad \rho = \frac{3M}{4\pi R^3}\ .$$

The moment of inertia is therefore

$$I = \int_V d^2(\mathbf{x})\ \rho(\mathbf{x})\ dV = \frac{3M}{4\pi R^3} \int_V (x^2 + y^2)\ dx\ dy\ dz\ .$$

Going to spherical polar coordinates, we have

$$\begin{aligned} I &= \frac{3M}{4\pi R^3} \int_0^R r^2\ dr \int_0^\pi \sin\theta\ d\theta \int_0^{2\pi} d\varphi\ r^2\ \sin^2\theta \\ &= \frac{3M}{4\pi R^3}\ 2\pi \int_0^R r^4\ dr \int_0^\pi \sin^3\theta\ d\theta = \frac{3M}{4\pi R^3}\ 2\pi\ \frac{R^5}{5}\ \frac{4}{3} = \frac{2MR^2}{5}\ . \end{aligned}$$

Example 2.11

Find the mass and the coordinates of the centre of mass of a semi-infinite sheet with surface mass density

$$\sigma(x, y) = \sigma_0\ e^{-(x^2+y^2)/a^2} \quad (\sigma_0\ ,\ a\ \text{constant})$$

in the positive x region of the xy plane ($\sigma = 0$ for $x < 0$).

The mass is

$$M = \sigma_0 \int_0^\infty dx\ e^{-x^2/a^2} \int_{-\infty}^\infty dy\ e^{-y^2/a^2} = \frac{\sigma_0 \pi a^2}{2}\quad ,$$

where we have used the result (2.20) for gaussian integration. The x coordinate of the centre of mass is

$$x_{\mathrm{CM}} = \frac{1}{\sigma_0 \pi a^2/2}\ \sigma_0 \int_0^\infty dx\ x\ e^{-x^2/a^2} \int_{-\infty}^\infty dy\ e^{-y^2/a^2} = \frac{a}{\sqrt{\pi}}\quad ,$$

where we have used again the gaussian integral. The y coordinate of the centre of mass is

$$y_{\mathrm{CM}} = \frac{1}{\sigma_0 \pi a^2/2}\ \sigma_0 \int_0^\infty dx\ e^{-x^2/a^2} \int_{-\infty}^\infty dy\ y\ e^{-y^2/a^2} = 0\quad ,$$

where we have used that the y integral vanishes because the integrand is antisymmetric.

3 Line integrals and surface integrals

In this section we introduce the integration of scalar and vector fields over curves and surfaces.

3.1 Line integrals

We begin by introducing line integrals, that is, integrals of fields over curves.

A curve γ in three-dimensional space is a vector valued function $\mathbf{x} = \mathbf{x}(t)$ of a real parameter t, defined in the interval $[a, b]$,

$$\mathbf{x}(t) = (x(t), y(t), z(t)) \quad , \qquad a \leq t \leq b \ . \tag{3.1}$$

Example 3.1

The straight line segment joining the origin $O = (0, 0, 0)$ with the point $P = (1, 1, 0)$ can be parameterized as

$$\begin{aligned} &x = t \\ &y = t \\ &z = 0 \\ &\text{with } \ 0 \leq t \leq 1 \ . \end{aligned} \tag{3.2}$$

The curve

$$\begin{aligned} &x = t^2 \\ &y = t^3 \\ &z = 0 \\ &\text{with } \ 0 \leq t \leq 1 \end{aligned} \tag{3.3}$$

joins the same two points O and P but along $y = x^{3/2}$.

Given the parameterization (3.1) of a curve γ, the first derivative

$$\mathbf{x}'(t) = \frac{d\mathbf{x}}{dt}$$

is the tangent vector to the curve. The arc length element along the curve can be written in terms of the function $\mathbf{x}(t)$ as

$$\begin{aligned} ds &= \sqrt{\mathbf{dx}^2} = \sqrt{dx^2 + dy^2 + dz^2} \\ &= dt\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} \ . \end{aligned}$$

The vector

$$\mathbf{t} = \frac{d\mathbf{x}}{ds}$$

is the unit vector tangent to the curve at each point.

A closed curve is a curve for which $\mathbf{x}(a) = \mathbf{x}(b)$ in Eq (3.1).

Example 3.2

The circle of radius 2 centred on the origin in the xy plane, $x^2 + y^2 = 4$, is represented in the parametric form (3.1) as

$$\begin{aligned} &x = 2\cos t \\ &y = 2\sin t \\ &z = 0 \\ &\text{for } \; 0 \le t \le 2\pi \; . \end{aligned}$$

Let us now consider a vector field $\mathbf{u}(\mathbf{x})$ defined in a region containing the curve γ, parameterized by the function $\mathbf{x}(t)$ as in Eq (3.1). We define the line integral of $\mathbf{u}$ along γ as

$$\int_\gamma \mathbf{u} \cdot \mathbf{dl} = \int_a^b \mathbf{u}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \; dt \; . \tag{3.4}$$

The meaning of the integration in Eq (3.4) can be understood along similar lines to the case of the integrals in Section 2. To this end, i) subdivide the curve into line elements $\mathbf{\Delta l}_i$, $i = 1, 2, \ldots, N$; ii) construct the sum

$$S = \sum_{i=1}^{N} \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{\Delta l}_i \; , \tag{3.5}$$

where $\mathbf{x}_i$ is a point on the curve belonging to the line element $\mathbf{\Delta l}_i$; iii) take the limit of the sum S in which all line elements become infinitesimal $|\mathbf{\Delta l}_i| \to 0$, that is, $N \to \infty$. It can be shown that this equals the line integral

$$\lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{\Delta l}_i = \int_\gamma \mathbf{u} \cdot \mathbf{dl} \; . \tag{3.6}$$

The line integral can be evaluated by decomposing the integrand into cartesian components as

$$\int_\gamma \mathbf{u} \cdot \mathbf{dl} = \int_\gamma (u_x dx + u_y dy + u_z dz) \; . \tag{3.7}$$

It measures the component of the field $\mathbf{u}$ along the direction tangent to the curve, multiplied by the arc length element and integrated over the whole curve.

Example 3.3

Evaluate the line integral of the vector field $\mathbf{u} = (\sqrt{y}, x^3 + y, 0)$ along the curve γ in Eq (3.2) and along the curve γ' in Eq (3.3).

For the curve γ in Eq (3.2) we have $\mathbf{x}'(t) = (1, 1, 0)$, so $\mathbf{u}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \sqrt{t} + t^3 + t$. Thus

$$\int_\gamma \mathbf{u} \cdot \mathbf{dl} = \int_0^1 (\sqrt{t} + t^3 + t) dt = 17/12 \; .$$

For the curve γ' in Eq (3.3) we have $\mathbf{x}'(t) = (2t, 3t^2, 0)$, so $\mathbf{u}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = 2t^{5/2} + 3t^8 + 3t^5$. Thus

$$\int_{\gamma'} \mathbf{u} \cdot \mathbf{dl} = \int_0^1 (2t^{5/2} + 3t^8 + 3t^5) dt = 59/42 \; .$$

Note:

If the curve γ is a closed curve the line integral is referred to as the circulation of vector field $\mathbf{u}$ around curve γ, and notated by

$$\oint_\gamma \mathbf{u} \cdot \mathbf{dl} = \text{circulation of } \mathbf{u} \text{ around } \gamma \; . \tag{3.8}$$

Line integrals have many applications in physics. An example is the concept of work. The work done by a force $\mathbf{F} = \mathbf{F}(\mathbf{x})$ on a particle undergoing a small displacement $\mathbf{dl}$ is $\mathbf{F} \cdot \mathbf{dl}$. If the particle moves along a trajectory from point A to point B represented by the curve γ, the work done by the force $\mathbf{F}$ is given by

$$W = \int_\gamma \mathbf{F} \cdot \mathbf{dl} \; ,$$

that is, the line integral of $\mathbf{F}$ along γ.

Example 3.4

Calculate the work done by the force field $\mathbf{F}(x, y) = \alpha(y^2, 2x(y + 1))$, where α is constant, on a particle moving in the xy plane along the straight line segment from $(1, 1)$ to $(2, 2)$.

$$W = \int_\gamma (F_x dx + F_y dy) = \alpha \int_1^2 (t^2 dt + 2t(t + 1) dt)$$
$$= \alpha \int_1^2 (3t^2 + 2t) dt = 10 \; \alpha \; .$$

The line integral of a scalar field $\phi(\mathbf{x})$ along a curve γ

$$\int_\gamma \phi \; dl \; ,$$

where dl is the arc length element, can be introduced by similar methods to the ones employed above.

Example 3.5

Evaluate the integral of the scalar field $\phi(x, y) = xy$ along the quartercircle C of radius 2 centred on the origin in the first quadrant of the xy plane.

Using plane polar coordinates, we have $x = 2\cos\varphi$, $y = 2\sin\varphi$. The arc length element is $dl = \sqrt{dx^2 + dy^2} = d\varphi\sqrt{(dx/d\varphi)^2 + (dy/d\varphi)^2} = 2d\varphi$. Then

$$\int_C \phi \; dl = \int_0^{\pi/2} 2 \; d\varphi (2\cos\varphi)(2\sin\varphi) = 8 \int_0^{\pi/2} \cos\varphi \sin\varphi = 4 \; .$$

3.2 Path independence and conservative fields

In general the line integral of a vector field $\mathbf{u}$ between two points A and B

$$\int_{\gamma(A\to B)} \mathbf{u}\cdot\mathbf{dl}$$

depends on the path γ to go from A to B.

There exists a special class of vector fields, however, for which the result is independent of the path and depends only on the initial and final points A and B. Such vector fields are referred to as conservative fields. For these fields, the line integral has the same value regardless of which path is taken from A to B, and the result can be written as the difference between the values which some scalar field ϕ takes at the initial and final points A and B,

$$\int_{\gamma_1^{(A\to B)}} \mathbf{u}\cdot\mathbf{dl} = \int_{\gamma_2^{(A\to B)}} \mathbf{u}\cdot\mathbf{dl} = \phi(B) - \phi(A) \tag{3.9}$$

for any curves γ_1, γ_2 joining A and B. Given the conservative field $\mathbf{u}$, the scalar field $\phi(\mathbf{x})$ in Eq (3.9) is called the "potential" of $\mathbf{u}$.

Let us analyze a few consequences of Eq (3.9).

i) If we take A and B to be two arbitrarily close points, Eq (3.9) can be written in infinitesimal form in terms of the differential $d\phi$ of the potential,

$$\mathbf{u}\cdot\mathbf{dx} = \phi(B) - \phi(A) = d\phi = \nabla\phi\cdot\mathbf{dx}\ , \tag{3.10}$$

and thus implies that

$$\mathbf{u} = \nabla\phi\ . \tag{3.11}$$

That is, conservative fields are gradients: a conservative field $\mathbf{u}$ can be written as the gradient of its potential ϕ.

ii) If we construct the closed curve C obtained by joining the curves γ_1 and $-\gamma_2$ in Eq (3.9), we see that the line integral of $\mathbf{u}$ along C must be zero

$$\oint_C \mathbf{u}\cdot\mathbf{dl} = 0\ \ . \tag{3.12}$$

This applies to any closed-curve integral of $\mathbf{u}$. Thus, the circulation of a conservative field is always zero.

iii) If the vector field $\mathbf{u}$ is conservative, its curl is zero. This can be seen by using Eq (3.11) and the theorem (1.16):

$$\nabla\wedge\mathbf{u} = \nabla\wedge\nabla\phi = 0\ \ . \tag{3.13}$$

Note:

Another way of stating the result given above in i) is to say that $\mathbf{u}\cdot\mathbf{dx}$ is an "exact differential". This means that $\mathbf{u}\cdot\mathbf{dx}$ has the form

$$\mathbf{u}\cdot\mathbf{dx} = \frac{\partial\phi}{\partial x}\ dx + \frac{\partial\phi}{\partial y}\ dy + \frac{\partial\phi}{\partial z}\ dz\ \ . \tag{3.14}$$

That is, the coefficient of each dx_i is the partial derivative of the same function ϕ with respect to x_i.

Note:

We will see in the next section that a converse of the result given above in iii) obtains: if vector field $\mathbf{u}$ has zero curl, under suitable hypotheses on the domain on which $\mathbf{u}$ is defined we may conclude that $\mathbf{u}$ is conservative, i.e., it is a gradient.

Example 3.6

Calculate the work done by the force field

$$\mathbf{F} = \lambda(2xy, x^2, 0) \ ,$$

where λ is a constant, on a particle traveling in the xy plane along the straight line segment from the origin $O = (0, 0, 0)$ to the point $P = (1, 1, 0)$.

On the particle's trajectory $y = x$, $dy = dx$. The work done by $\mathbf{F}$ is given by

$$W = \int_{\gamma(O \to P)} \mathbf{F} \cdot \mathbf{dl} = \lambda \int_0^1 (2x^2 dx + x^2 dx) = \lambda \left(x^3\right)_0^1 = \lambda \ .$$

Example 3.7

What is the work done by the force $\mathbf{F}$ in the previous example if the particle travels from O to P along the path made of a straight-line segment from O to $P' = (0, 1, 0)$ followed by a straight-line segment from P' to P?

The force $\mathbf{F}$ is conservative: $\mathbf{F} = \nabla(\lambda x^2 y)$. So the work done along this path is the same as the work along the path in the previous example, $W = \lambda$.

Example 3.8

Suppose the force field of the previous two examples is replaced by the force field

$$\mathbf{G} = \lambda(x^2, 2xy, 0) \ .$$

Calculate the work done by $\mathbf{G}$ on a particle traveling along the two paths from O to P in the previous two examples.

The force field $\mathbf{G}$ is not conservative: $\nabla \wedge \mathbf{G} = \lambda(0, 0, 2y) \neq 0$. So the work done by $\mathbf{G}$ is in general path-dependent. The work on the straight line segment γ from O to P is

$$W_1 = \int_{\gamma(O \to P)} \mathbf{G} \cdot \mathbf{dl} = \lambda \int_0^1 (x^2 dx + 2x^2 dx) = \lambda \left(x^3\right)_0^1 = \lambda \ ,$$

while the work on the path Γ made of the two straight line segments from O to P' and from P' to P is

$$W_2 = \int_{\Gamma(O \to P)} \mathbf{G} \cdot \mathbf{dl} = \lambda \int_0^1 0 \ dy + \lambda \int_0^1 x^2 dx = \lambda \left(\frac{x^3}{3}\right)_0^1 = \frac{\lambda}{3} \ .$$

3.3 Surface integrals

We now introduce the integration of fields over surfaces.

A surface Σ in three-dimensional space is a vector valued function $\mathbf{x} = \mathbf{x}(u, v)$ of two real parameters u and v, defined over a region R,

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \ , \quad \text{with} \ (u, v) \ \text{in} \ R \ . \tag{3.15}$$

This is analogous to the representation (3.1) of a curve. To parameterize a surface though we need two parameters instead of one. Equivalently, a surface may be represented by equations $z = f(x, y)$, or $F(x, y, z) = 0$.

Given the parameterization (3.15) of a surface Σ, the first derivatives

$$\frac{\partial \mathbf{x}}{du} \ , \ \frac{\partial \mathbf{x}}{dv}$$

are the tangent vectors to the surface. Their cross product

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v}$$

is the normal vector to the surface. The unit vector normal to the surface at each point is obtained as

$$\mathbf{n} = \frac{\partial \mathbf{x}/\partial u \wedge \partial \mathbf{x}/\partial v}{|\partial \mathbf{x}/\partial u \wedge \partial \mathbf{x}/\partial v|} \ . \tag{3.16}$$

The area element dS at any point on the surface is given by the area of the parallelogram constructed from infinitesimal du and dv displacements on the surface,

$$dS = \left| \frac{\partial \mathbf{x}}{\partial u} \ du \wedge \frac{\partial \mathbf{x}}{\partial v} \ dv \right| = \left| \frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \right| du \ dv \ . \tag{3.17}$$

Using Eqs (3.16), (3.17) we define

$$\mathbf{dS} = \mathbf{n} \ dS \ .$$

Example 3.9

The spherical surface of radius R centred on the origin, $x^2 + y^2 + z^2 = R^2$, has the parametric representation in terms of two real parameters $u = \theta$ and $v = \varphi$

$$\begin{aligned} &x = R \sin\theta \cos\varphi \\ &y = R \sin\theta \sin\varphi \\ &z = R \cos\theta \\ &\text{with} \ 0 \leq \theta \leq \pi \ , \ 0 \leq \varphi \leq 2\pi \ . \end{aligned}$$

From

$$\frac{\partial \mathbf{x}}{\partial \theta} = (R \cos\theta \cos\varphi, R \cos\theta \sin\varphi, -R \sin\theta) \ ,$$

$$\frac{\partial \mathbf{x}}{\partial \varphi} = (-R\sin\theta\sin\varphi, R\sin\theta\cos\varphi, 0)$$

we get

$$\frac{\partial \mathbf{x}}{\partial \theta} \wedge \frac{\partial \mathbf{x}}{\partial \varphi} = R^2 \sin\theta \; \widehat{\mathbf{x}} \ ,$$

where $\widehat{\mathbf{x}}$ is the unit vector in the radial direction. So $\mathbf{n} = \widehat{\mathbf{x}}$, $dS = R^2 \sin\theta \; d\theta \; d\varphi$.

We set the orientation of the surface as follows. For a closed surface, we take the outward normal to the surface. For an open surface, we orient the boundary curve and set the normal to the surface from the right hand rule.

Let us now consider a vector field $\mathbf{u}(\mathbf{x})$ defined in a region containing the surface Σ, parameterized by the function $\mathbf{x}(u, v)$ as in Eq (3.15). We define the surface integral of $\mathbf{u}$ over Σ as

$$\begin{aligned} \int_\Sigma \mathbf{u} \cdot \mathbf{dS} &= \int_R \mathbf{u}(\mathbf{x}(u,v)) \cdot \mathbf{n} \left| \frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \right| \; du \; dv \\ &= \int_R \mathbf{u}(\mathbf{x}(u,v)) \cdot \mathbf{n} \; dS \ . \end{aligned} \tag{3.18}$$

The meaning of the surface integral in Eq (3.18) can be understood analogously to the case of the line integral. To this end, i) subdivide the surface Σ into area elements $\mathbf{\Delta S}_i$, $i = 1, 2, \ldots, N$; ii) construct the sum

$$S = \sum_{i=1}^{N} \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{\Delta S}_i \ \ , \tag{3.19}$$

where $\mathbf{x}_i$ is a point on the surface in the area element $\mathbf{\Delta S}_i$; iii) take the limit of the sum S in which all area elements become infinitesimal, $|\mathbf{\Delta S}_i| \to 0$, i.e., $N \to \infty$. It can be shown that this equals the surface integral

$$\lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{\Delta S}_i = \int_\Sigma \mathbf{u} \cdot \mathbf{dS} \ . \tag{3.20}$$

The integral (3.18) is referred to as the "flux" of the vector field $\mathbf{u}$ across the surface Σ,

$$\int_\Sigma \mathbf{u} \cdot \mathbf{dS} = \text{flux of } \mathbf{u} \text{ across } \Sigma \ . \tag{3.21}$$

The flux measures the component of the field $\mathbf{u}$ along the direction normal to the surface, multiplied by the area element and integrated over the whole surface.

Example 3.10

Calculate the flux of the uniform field

$$\mathbf{u} = u_0(0, 0, 1) \ ,$$

where u_0 is constant, through the hemispherical surface $x^2 + y^2 + z^2 = R^2$, $z \geq 0$.

Going to spherical polar coordinates, we have $dS = R^2 \sin\theta d\theta d\varphi$, and $\mathbf{u}\cdot\mathbf{n} = u_0 \cos\theta$. So the flux is given by the surface integral

$$\begin{aligned}\int_{\text{hemisphere}} \mathbf{u}\cdot\mathbf{dS} &= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi\ R^2 \sin\theta\ u_0 \cos\theta \\ &= 2\pi u_0 R^2 \left(-\frac{\cos 2\theta}{4}\right)_0^{\pi/2} = \pi u_0 R^2\end{aligned}$$

Example 3.11

Calculate the integral

$$I = \int_\Sigma \mathbf{u}\cdot\mathbf{dS}$$

where $\mathbf{u}$ is the vector field $\mathbf{u} = (z, x, -3y^2 z)$ and Σ is the portion of the cylindrical surface in the first octant $x^2 + y^2 = 16$, $0 \le z \le 5$, $x \ge 0$, $y \ge 0$.

Going to cylindrical polar coordinates, we have $dS = Rd\varphi dz$, and $\mathbf{u}\cdot\mathbf{n} = z\cos\varphi + R\cos\varphi\sin\varphi$, with $R = 4$. Then

$$\begin{aligned}I &= \int_0^5 dz \int_0^{\pi/2} 4\ d\varphi\ (z\cos\varphi + 4\cos\varphi\sin\varphi) \\ &= 4\int_0^5 dz\,(z+2) = 90\end{aligned}$$

Surface integrals have many applications in physics. An example is fluid dynamics. Consider a surface Σ in a fluid which has density $\rho(\mathbf{x})$ and moves with velocity field $\mathbf{v}(\mathbf{x})$. The mass of fluid which crosses an element of surface with area dS and normal $\mathbf{n}$ in time dt is

$$dm = \rho\ \mathbf{v}\cdot\mathbf{n}\ dS\ dt\ .$$

So the total mass which flows through the surface Σ per unit time is given by

$$\begin{aligned}\frac{dM}{dt} &= \int_\Sigma \rho\ \mathbf{v}\cdot\mathbf{n}\ dS \\ &= \int_\Sigma \mathbf{J}\cdot\mathbf{dS}\ ,\end{aligned}$$

where $\mathbf{J} \equiv \rho\mathbf{v}$. That is, it is given by the flux of $\rho\mathbf{v}$ across Σ.

Note:

If Σ is a closed surface, we will denote the flux of the vector field $\mathbf{u}$ across Σ by the integral symbol

$$\oint_\Sigma \mathbf{u}\cdot\mathbf{dS}\ .$$

4 Divergence theorem and Stokes' theorem

There exist integral theorems in vector analysis which establish general relations between line, surface and volume integrals. We here study two of the most important ones, the divergence theorem and Stokes' theorem, and discuss some of their corollaries and applications.

4.1 Divergence theorem

Let $\mathbf{u} = \mathbf{u}\,(x, y, z)$ be a vector field, continuous, differentiable and with continuous first derivatives in a region containing a volume V. Let V be bounded by the closed surface ∂V. Then

$$\oint_{\partial V} \mathbf{u} \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{u}\, dV\ . \tag{4.1}$$

Eq (4.1) is the divergence theorem. It states that the flux of a vector field through a closed surface equals the volume integral of the divergence of the field over the volume bounded by the surface.

The divergence theorem (4.1) can be derived by i) subdividing the volume V into infinitesimal parallelepipeds of sides Δx, Δy, Δz, ii) evaluating the flux through the boundary surface of each parallelepiped, iii) summing over all parallelepipeds and taking the limit in which their sides become vanishingly small.

Consider point $P = (x, y, z)$ in volume V and a parallelepiped with vertex in P and sides Δx, Δy, Δz, of volume $\Delta V = \Delta x\ \Delta y\ \Delta z$ and boundary surface ΔS. The flux of $\mathbf{u}$ out of this parallelepiped can be written as the sum of the fluxes through its six faces. This sum in turn can be arranged in terms of three pairs of parallel faces as

$$\begin{aligned}
\oint_{\Delta S} \mathbf{u} \cdot \mathbf{dS} &= [u_x(x + \Delta x, y, z) - u_x(x, y, z)]\Delta y\ \Delta z \\
&+ [u_y(x, y + \Delta y, z) - u_y(x, y, z)]\Delta x\ \Delta z \\
&+ [u_z(x, y, z + \Delta z) - u_z(x, y, z)]\Delta x\ \Delta y \\
&\simeq \frac{\partial u_x}{\partial x}\ \Delta x\ \Delta y\ \Delta z + \frac{\partial u_y}{\partial y}\ \Delta y\ \Delta x\ \Delta z + \frac{\partial u_z}{\partial z}\ \Delta z\ \Delta x\ \Delta y \\
&= \nabla \cdot \mathbf{u}\ \Delta V\ .
\end{aligned} \tag{4.2}$$

Note that the signs in each term of the three lines after the first equality take into account that the fluxes are defined with the outward normal to the surface. In the next-to-last line we have used that Δx, Δy, Δz are small to rewrite the differences of $\mathbf{u}$ field components at small spatial separations in terms of first derivatives. Eq (4.2) recasts the flux of $\mathbf{u}$ out of the parallelepiped in terms of the divergence of $\mathbf{u}$ at point P times the volume ΔV.

Now let us construct the sum of the fluxes over all the parallelepipeds of volume ΔV_i and boundary surface ΔS_i, with $i = 1, 2, \ldots, N$, into which the volume V is subdivided, and let us take the limit Δx, Δy, $\Delta z \to 0$, i.e., $N \to \infty$. Using the result (4.2), we have

$$\lim_{N\to\infty} \sum_{i=1}^{N} \oint_{\Delta S_i} \mathbf{u} \cdot \mathbf{dS} = \lim_{N\to\infty} \sum_{i=1}^{N} \nabla \cdot \mathbf{u}\, \Delta V_i \ . \tag{4.3}$$

In the sum of surface integrals on the left hand side of Eq (4.3), we note that any given face belonging to neighboring parallelepipeds will give two equal and opposite contributions to the sum — with the opposite sign due to the fact that fluxes are defined with the outward normal to the surface in the case of each parallelepiped. Thus all terms in the left hand side of Eq (4.3) will cancel pairwise, except those corresponding to sides of parallelepipeds lying on the boundary ∂V of the volume — which only appear once in the sum and do not have anything to cancel against. In the limit $N \to \infty$, therefore, the sum on the left hand side of Eq (4.3) gives the surface integral of $\mathbf{u}$ over the boundary of V.

On the other hand, the sum on the right hand side of Eq (4.3) in the limit $N \to \infty$ gives the volume integral of $\nabla \cdot \mathbf{u}$. Thus we obtain

$$\oint_{\partial V} \mathbf{u} \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{u}\, dV \ ,$$

which is Eq (4.1).

Note:

Eq (4.2) is the infinitesimal version of the divergence theorem. For $\Delta V \to 0$ it can be viewed as providing a definition, alternative to that in Eq (1.9), of the divergence of a vector field $\mathbf{u}$ at any given point P,

$$\nabla \cdot \mathbf{u} = \lim_{\Delta V\to 0} \frac{1}{\Delta V} \oint_{\Delta S} \mathbf{u} \cdot \mathbf{dS} \ . \tag{4.4}$$

Eq (4.4) defines the divergence of $\mathbf{u}$ as the outward flux ("flow") of $\mathbf{u}$ per unit volume, in a local manner independent of the coordinate system.

Example 4.1

Given the vector field $\mathbf{F} = (4x, -2y^2, z^2)$, verify the divergence theorem for the volume V enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$.

To evaluate the left hand side of Eq (4.1), we sum the three surface integrals across the top, the bottom and the side of the cylinder:

$$\oint_{\partial V} \mathbf{F} \cdot \mathbf{dS} = \int_{\text{top}} \mathbf{F} \cdot \mathbf{dS} + \int_{\text{bottom}} \mathbf{F} \cdot \mathbf{dS} + \int_{\text{side}} \mathbf{F} \cdot \mathbf{dS} \ .$$

On the top $\mathbf{F} \cdot \mathbf{dS} = 9\ dS$, so

$$\int_{\text{top}} \mathbf{F} \cdot \mathbf{dS} = 9 \times 2^2 \pi = 36\pi \ .$$

At the bottom $\mathbf{F} \cdot \mathbf{dS} = 0$, so

$$\int_{\text{bottom}} \mathbf{F} \cdot \mathbf{dS} = 0 \; .$$

On the side, using cylindrical polars, $\mathbf{F} \cdot \mathbf{dS} = (8\cos^2\varphi - 8\sin^3\varphi)\ 2\ d\varphi\ dz$, so

$$\int_{\text{side}} \mathbf{F} \cdot \mathbf{dS} = \int_0^{2\pi} 2\ d\varphi \int_0^3 dz\ (8\cos^2\varphi - 8\sin^3\varphi) = 48\pi \; .$$

Thus

$$\oint_{\partial V} \mathbf{F} \cdot \mathbf{dS} = 36\pi + 48\pi = 84\pi \; . \tag{4.5}$$

For the right hand side of Eq (4.1) we have $\nabla \cdot \mathbf{F} = 4 - 4y + 2z$. Using cylindrical polar coordinates, the volume integral of the divergence of $\mathbf{F}$ is given by

$$\begin{aligned} \int_V \nabla \cdot \mathbf{F}\ dV &= \int_0^{2\pi} d\varphi \int_0^2 r\ dr \int_0^3 dz\ (4 - 4r\sin\varphi + 2z) \\ &= \int_0^{2\pi} d\varphi \int_0^2 r\ dr\ (12 - 12r\sin\varphi + 9) \\ &= \int_0^{2\pi} d\varphi\ (42 - 32\sin\varphi) = 84\pi \; . \end{aligned} \tag{4.6}$$

This equals the result (4.5), so the divergence theorem is verified.

Note:

A corollary of the divergence theorem is that the flux of any curl through any closed surface is zero. That is,

$$\oint_\Sigma \nabla \wedge \mathbf{F} \cdot \mathbf{dS} = 0 \tag{4.7}$$

for any vector field $\mathbf{F}$ and closed surface Σ. This is because by divergence theorem

$$\oint_\Sigma \nabla \wedge \mathbf{F} \cdot \mathbf{dS} = \int_V \nabla \cdot \nabla \wedge \mathbf{F}\ dV$$

which vanishes since $\nabla \cdot \nabla \wedge \mathbf{F} = 0$ (Eq (1.17)).

Example 4.2

Maxwell's equations of electromagnetism can equivalently be formulated in differential form or in integral form. The divergence theorem can be used to relate the two formulations in the case of Maxwell's equations for the divergence of electromagnetic fields.

For example, the differential form of Gauss' law is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} , \tag{4.8}$$

where $\mathbf{E}$ is the electric field, ρ is the electric charge density, and ε_0 is a physical constant, the electric permittivity of vacuum. Use the divergence theorem to obtain the equivalent, integral form of Gauss' law.

We proceed as follows. Let us integrate Eq (4.8) over a volume V. We thus get

$$\int_V \nabla \cdot \mathbf{E} \, dV = \frac{1}{\varepsilon_0} \int_V \rho \, dV .$$

By the divergence theorem, the left hand side equals the flux of the electric field through the closed surface ∂V bounding volume V. The integral over the charge density ρ on the right hand side gives the total electric charge Q_V contained in volume V. Therefore

$$\oint_{\partial V} \mathbf{E} \cdot \mathbf{dS} = \frac{Q_V}{\varepsilon_0} . \tag{4.9}$$

Eq (4.9) is the integral form of Gauss' law, equivalent to the differential form (4.8). It states that the flux of the electric field through any closed surface ∂V equals the electric charge enclosed by that surface divided by ε_0.

Note:

Analogously, Maxwell's equation in differential form for the divergence of the magnetic field $\mathbf{B}$,

$$\nabla \cdot \mathbf{B} = 0 , \tag{4.10}$$

can equivalently be formulated in integral form, by using the divergence theorem, as

$$\oint_{\partial V} \mathbf{B} \cdot \mathbf{dS} = 0 , \tag{4.11}$$

that is, the flux of the magnetic field vanishes through any closed surface ∂V.

Example 4.3

Among the most important physical applications of the divergence theorem is the continuity equation. Consider a fluid, with mass per unit volume ρ, flowing out of a given volume V. Let the mass flowing out per unit area perpendicular to direction $\mathbf{n}$ and per unit time be $\mathbf{J} \cdot \mathbf{n}$. Show that the principle of mass conservation requires

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 . \tag{4.12}$$

The conservation of mass implies that the rate of change of the mass M contained in the volume V must equal the net rate at which mass is entering or leaving the volume,

$$\frac{dM}{dt} = - \oint_{\partial V} \mathbf{J} \cdot \mathbf{dS} ,$$

where on the right hand side is the integral of $\mathbf{J}$ over the closed surface bounding V, and the minus sign is because in the integral we take the outward normal to the surface. Now the mass M can be expressed as the volume integral of ρ over V, so that

$$\frac{d}{dt}\int_V \rho \, dV = -\oint_{\partial V} \mathbf{J}\cdot\mathbf{dS} \, . \tag{4.13}$$

By bringing the time derivative inside the integral on the left hand side and applying the divergence theorem to the right hand side, we get

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J}\right) \, dV = 0 \, .$$

But this must apply to any volume V. Then we must have

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \, ,$$

which is Eq (4.12). This is referred to as the continuity equation. Eq (4.13) is the integral form of the continuity equation.

In the above discussion ρ is the density (mass per unit volume) and $\mathbf{J}$ is the flux density (mass flowing through unit transverse area per unit time). In electromagnetism, the same equation (4.12) obtains (or (4.13)) to express the conservation of electric charge, with ρ the electric charge density and $\mathbf{J}$ the electric current density. A continuity equation of the same form also holds to express the conservation of energy in continuous systems, with ρ being replaced by the energy density (energy per unit volume) and $\mathbf{J}$ by the energy density flux (energy flowing through unit transverse area per unit time). A continuity equation of the same form holds in quantum mechanics too to express the principle of probability conservation, with ρ representing the probability density and $\mathbf{J}$ the probability current density.

4.2 Stokes' theorem

Let $\mathbf{u} = \mathbf{u}\ (x, y, z)$ be a continuously differentiable vector field in a region containing an oriented surface Σ. Let Σ be bounded by the closed curve $\partial\Sigma$. Then

$$\oint_{\partial\Sigma} \mathbf{u}\cdot\mathbf{dl} = \int_\Sigma \nabla\wedge\mathbf{u}\cdot\mathbf{dS} \, . \tag{4.14}$$

Eq (4.14) is the Stokes' theorem. It states that the circulation of a vector field around a closed curve equals the flux of the curl of the field through a surface bounded by the curve.

Note:

Given a closed curve, Stokes' theorem applies to any surface bounded by the curve. That is, for any two surfaces Σ_1 and Σ_2 having the same boundary curve $\partial\Sigma$ we have

$$\oint_{\partial\Sigma} \mathbf{u} \cdot \mathbf{dl} = \int_{\Sigma_1} \nabla \wedge \mathbf{u} \cdot \mathbf{dS} = \int_{\Sigma_2} \nabla \wedge \mathbf{u} \cdot \mathbf{dS} \, . \tag{4.15}$$

The last equality is a consequence of the corollary (4.7) of the divergence theorem: by applying this to the closed surface obtained by joining Σ_1 and Σ_2, one has

$$\oint_{\Sigma_1 - \Sigma_2} \nabla \wedge \mathbf{u} \cdot \mathbf{dS} = 0 \;\Rightarrow\; \int_{\Sigma_1} \nabla \wedge \mathbf{u} \cdot \mathbf{dS} = \int_{\Sigma_2} \nabla \wedge \mathbf{u} \cdot \mathbf{dS} \, ,$$

where the signs for Σ_1 and Σ_2 take into account the orientation of the surfaces.

Stokes' theorem (4.14) can be derived by i) tiling the surface Σ with infinitesimal plaquettes, which may be oriented along directions x and y, with sides Δx and Δy, ii) evaluating the circulation around the boundary curve of each plaquette, iii) summing over all plaquettes and taking the limit in which their sides become vanishingly small.

Consider point $P = (x, y, z)$ on the surface Σ and a plaquette with vertex in P and sides Δx and Δy, with area $\Delta S = \Delta x \; \Delta y$ and boundary curve ΔC. The circulation of $\mathbf{u}$ around ΔC can be written as the sum of the line integrals along the four sides of the plaquette. This sum in turn can be arranged in terms of two pairs of parallel sides as

$$\begin{aligned} \oint_{\Delta C} \mathbf{u} \cdot \mathbf{dl} &= [u_x(x, y, z) - u_x(x, y + \Delta y, z)]\Delta x \\ &\quad + [u_y(x + \Delta x, y, z) - u_y(x, y, z)]\Delta y \\ &\simeq -\frac{\partial u_x}{\partial y} \; \Delta y \; \Delta x + \frac{\partial u_y}{\partial x} \; \Delta x \; \Delta y \\ &= \nabla \wedge \mathbf{u} \cdot \mathbf{\Delta S} \, . \end{aligned} \tag{4.16}$$

Here the signs in each term of the two lines after the first equality take into account the direction of circulation around ΔC, dictated by the orientation of the plaquette. In the next-to-last line we have used that Δx and Δy are small to rewrite the differences of $\mathbf{u}$ field components at small spatial separations in terms of first derivatives. In the last line we have rewritten, in a coordinate-independent fashion, the circulation of $\mathbf{u}$ around the boundary curve of the plaquette in terms of the flux of the curl of $\mathbf{u}$ through the plaquette, where $\mathbf{\Delta S} = \mathbf{n} \; \Delta S$, with $\mathbf{n}$ the unit vector normal to the plaquette.

Now let us construct the sum of the line integrals over all the plaquettes of area ΔS_i and boundary curve ΔC_i, for $i = 1, 2, \ldots, N$, with which we have tiled the surface Σ, and let us take the limit Δx, $\Delta y \to 0$, i.e., $N \to \infty$. Using the result (4.16), we have

$$\lim_{N\to\infty} \sum_{i=1}^{N} \oint_{\Delta C_i} \mathbf{u} \cdot \mathbf{dl} = \lim_{N\to\infty} \sum_{i=1}^{N} \nabla \wedge \mathbf{u} \cdot \mathbf{\Delta S}_i \, . \tag{4.17}$$

In the sum of line integrals on the left hand side of Eq (4.17), we note that any given side belonging to neighboring plaquettes will contribute two equal and opposite terms to the sum — with the opposite sign due to the fact that the given side is travelled in opposite directions in the case of the two neighboring plaquettes. Thus all terms in the left hand side of Eq (4.17) will cancel pairwise, except those corresponding to sides of plaquettes lying on the boundary $\partial\Sigma$ of the surface — which only appear once in the sum and do not have anything to cancel against. In the limit $N \to \infty$, therefore, the sum on the left hand side of Eq (4.17) gives the line integral of $\mathbf{u}$ over the boundary of Σ.

On the other hand, the sum on the right hand side of Eq (4.17) in the limit $N \to \infty$ gives the surface integral of $\nabla \wedge \mathbf{u}$ over the surface Σ. Thus we obtain

$$\oint_{\partial\Sigma} \mathbf{u} \cdot \mathbf{dl} = \int_{\Sigma} \nabla \wedge \mathbf{u} \cdot \mathbf{dS} \ ,$$

which is Eq (4.14).

Note:

Eq (4.16) is the infinitesimal version of Stokes' theorem. For $\Delta S \to 0$ it can be viewed as providing a definition, alternative to that in Eq (1.10), of the curl of a vector field $\mathbf{u}$ at any given point P,

$$\nabla \wedge \mathbf{u} \cdot \mathbf{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{\Delta C} \mathbf{u} \cdot \mathbf{dl} \ . \tag{4.18}$$

Here ΔS is the area of the infinitesimal surface at P, $\mathbf{n}$ is the unit vector normal to the surface, ΔC is the boundary of the surface. Eq (4.18) defines the curl of $\mathbf{u}$ as the circulation ("rotation") of $\mathbf{u}$ per unit area, in a local manner independent of the coordinate system.

Example 4.4

Given the vector field $\mathbf{F} = (y, -x, z)$, verify Stokes' theorem for the hemispherical surface $x^2 + y^2 + z^2 = 3$, $z \geq 0$.

The left hand side of Eq (4.14) is given by the line integral of $\mathbf{F}$ around the boundary curve of the hemispherical surface, which is the circle $x^2 + y^2 = 3$ in the xy plane. Using plane polar coordinates, this is

$$\oint_{\partial\Sigma} \mathbf{F} \cdot \mathbf{dl} = \int_0^{2\pi} (-\sqrt{3}) \ \sqrt{3} \ d\theta = -6\pi \ . \tag{4.19}$$

In the right hand side of Eq (4.14), the curl of $\mathbf{F}$ is given by $\nabla \wedge \mathbf{F} = -2\mathbf{k}$. Using spherical polar coordinates, the flux of the curl of $\mathbf{F}$ is

$$\begin{aligned} \int_{\Sigma} \nabla \wedge \mathbf{F} \cdot \mathbf{dS} &= \int_0^{2\pi} d\varphi \int_0^{\pi/2} (-2\cos\theta) \ 3 \ \sin\theta \ d\theta \\ &= -6(2\pi)(1/2) = -6\pi \ . \end{aligned} \tag{4.20}$$

This equals the result (4.19), so Stokes' theorem is verified.

Note:

In Section 3.2, around Eq (3.13), we have seen that if a vector field $\mathbf{u}$ is conservative, its curl is zero. Stokes' theorem provides us with a converse of this. To see this, note that if $\nabla \wedge \mathbf{u} = 0$ then the line integral of $\mathbf{u}$ around any closed path, by applying Stokes' theorem, is zero. This in turn implies that the line integral of $\mathbf{u}$ between any two points A and B is path-independent, that is, using an argument similar to that around Eq (3.10), $\mathbf{u} = \nabla\phi$. So $\mathbf{u}$ is conservative. An important caveat is that for this reasoning to apply the condition $\nabla \wedge \mathbf{u} = 0$ must be verified everywhere inside any closed curve C. This can be expressed by saying that it must be verified on a "simply connected" region, i.e., a region "with no holes".

Example 4.5

Green's theorem in the plane. Applying Stokes' theorem to the two-dimensional case of a region R in the plane yields Green's theorem. Let R be a region in the xy plane, bounded by the closed curve ∂R, and let $\mathbf{F}$ be a vector field on R. The projection of $\nabla \wedge \mathbf{F}$ on the normal to the plane is $\partial F_y/\partial x - \partial F_x/\partial y$. Thus, Stokes' theorem (4.14) gives

$$\oint_{\partial R} (F_x dx + F_y dy) = \int_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx\ dy\ . \tag{4.21}$$

Eq (4.21) is Green's theorem.

Note:

Green's theorem implies that the area A of the region surrounded by a closed curve C in the plane can be computed by evaluating a line integral around C. To this end, take the vector field $\mathbf{F} = (-y, x)$. Then Green's theorem gives

$$\oint_C (xdy - ydx) = \int (1+1)\, dx\ dy$$

$$\Rightarrow\ A = \frac{1}{2} \oint_C (xdy - ydx)\ . \tag{4.22}$$

Example 4.6

Calculate the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

In plane polar coordinates we have $x = a\cos\varphi$, $y = b\sin\varphi$. Using Eq (4.22), the area is given by

$$A = \frac{1}{2} \int_0^{2\pi} d\varphi\ ab(\cos^2\varphi + \sin^2\varphi) = \frac{1}{2}\ ab\ (2\pi) = \pi ab\ .$$

Note:

The divergence theorem (4.1) and Stokes' theorem (4.14) have a similar structure as they express the integral of some derivative of a function over a given set S in terms of the function on a set of lower dimension, the boundary of S,

$$\int_S dF = \int_{\partial S} F \ , \tag{4.23}$$

where we have symbolically denoted by F and dF the function and its derivative. Eq (4.23) can be given a precise meaning by appropriate definition of differentials dF.

Example 4.7

Stokes' theorem can be used to relate the differential and integral formulations of Maxwell's equations for the curl of electromagnetic fields.

For example, the differential form of Faraday's law is

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ , \tag{4.24}$$

where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field. Use Stokes' theorem to obtain the equivalent, integral form of Faraday's law.

We proceed as follows. Let us integrate Eq (4.24) over an oriented surface Σ and evaluate the flux through Σ of the left hand side and right hand side. We get

$$\int_\Sigma \nabla \wedge \mathbf{E} \cdot \mathbf{dS} = -\int_\Sigma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} \ .$$

By Stokes' theorem, the left hand side equals the circulation of the electric field around the closed curve $\partial\Sigma$ bounding the surface Σ. The integral on the right hand side gives the time derivative of the flux of the magnetic field through Σ. Therefore

$$\oint_{\partial\Sigma} \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt} \int_\Sigma \mathbf{B} \cdot \mathbf{dS} \ . \tag{4.25}$$

Eq (4.25) is the integral form of Faraday's law, equivalent to the differential form (4.24). It states that the circulation of the electric field around any closed curve $\partial\Sigma$ equals minus the rate of change of the magnetic flux through the surface Σ bounded by the closed curve.

Note that Eq (4.25) applies to *any* surface having boundary curve $\partial\Sigma$. This is similar to the argument given for Eq (4.15): for two surfaces Σ_1 and Σ_2 having the same boundary curve $\partial\Sigma$ we have

$$\int_{\Sigma_1} \mathbf{B} \cdot \mathbf{dS} = \int_{\Sigma_2} \mathbf{B} \cdot \mathbf{dS}$$

because, from Eq (4.11), the flux of the magnetic field vanishes through the closed surface obtained by joining Σ_1 and Σ_2.