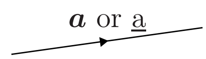
1 Revision

Reading: Chapter 14 Vectors (pp 367-421)



A vector is a quantity with magnitude and direction. In these notes, I will use bold letters to indicate vectors, for example $\boldsymbol{a}$ and $\boldsymbol{F}$. On the board, I will write these as $\underline{a}$ and $\underline{F}$.

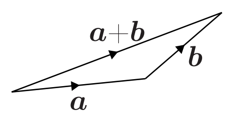
1.1 Components of a Vector

In three dimensions, any vector can be expressed as a linear combination of three orthogonal basis vectors. In cartesian coordinates, these are the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. We write

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}. \tag{1.1}$$

The coefficients a_x, a_y and a_z are called the components of the vector. I will sometimes use the notation $\boldsymbol{a} = (a_x, a_y, a_z)$. Note that, if we change our choice of basis vectors, the components will be different.

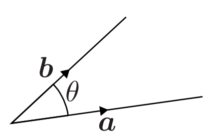
1.2 Adding Vectors



The sum of two vectors is also a vector. We add vectors using a vector triangle. In component form, we do the addition by adding the corresponding components. If $\boldsymbol{a} = (a_x, a_y, a_z)$ and $\boldsymbol{b} = (b_x, b_y, b_z)$, their sum is

$$\boldsymbol{a} + \boldsymbol{b} = (a_x + b_x)\boldsymbol{i} + (a_y + b_y)\boldsymbol{j} + (a_z + b_z)\boldsymbol{k}. \tag{1.2}$$

1.3 Scalar Product



The scalar product of two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, is a scalar which has magnitude $ab\cos\theta$, where a and b are the lengths of the vectors and θ is the angle between them. In component form it is

$$\boldsymbol{a}.\boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z. \tag{1.3}$$

1.4 Vector Product

The vector product of two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, is a vector which is perpendicular to the plane defined by $\boldsymbol{a}$ and $\boldsymbol{b}$ and has magnitude $ab\sin\theta$. In component form, it is given by

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{1.4}$$

$$= (a_y b_z - a_z b_y)\boldsymbol{i} + (a_z b_x - a_x b_z)\boldsymbol{j} + (a_x b_y - a_y b_x)\boldsymbol{k}. \tag{1.5}$$

Note that

$$\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}. \tag{1.6}$$

1.5 Triple Scalar Product

The triple scalar product is $\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$. It is a scalar quantity, and in component form it is most easily written as a determinant:

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{1.7}$$

The triple scalar product satisfies the cyclic relationship

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c}.(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b}.(\boldsymbol{c} \times \boldsymbol{a}) \tag{1.8}$$

1.6 Triple Vector Product

The triple vector product is $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$. It is a vector quantity which satisfies

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}.\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}.\boldsymbol{b}) \tag{1.9}$$

This vector lies is in the plane defined by $\boldsymbol{b}$ and $\boldsymbol{c}$. This makes sense, because $\boldsymbol{b} \times \boldsymbol{c}$ is perpendicular to this plane, so the cross product with another vector, $\boldsymbol{a}$, must lie in the plane.

1.6.1 Proof

Let us start by considering the $x-$component of the left hand side, $[\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})]_x$. In component form, we have

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ (b_y c_z - b_z c_y) & (b_z c_x - b_x c_z) & (b_x c_y - b_y c_x) \end{vmatrix}.$$

Hence

$$[\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})]_x = a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z) = b_x(a_y c_y + a_z b_z) - c_x(a_y b_y + a_z b_z).$$

Adding and subtracting $a_x b_x c_x$ gives

$$[\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})]_x = b_x(a_x c_x + a_y c_y + a_z b_z) - c_x(a_x b_x + a_y b_y + a_z b_z) = b_x(\boldsymbol{a}.\boldsymbol{c}) - c_x(\boldsymbol{a}.\boldsymbol{b}),$$

which is just the $x-$component of the right hand side of Eq.(1.9). From symmetry, the other components will also be equal, so the result is proved.

Example

By evaluating both sides of Eq.(1.9) for the vectors

$$\boldsymbol{a} = (2, -2, 3) \qquad \boldsymbol{b} = (1, -3, 4) \qquad \boldsymbol{c} = (0, -3, 2),$$

show that they are equal.

We have

$$\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -3 & 4 \\ 0 & -3 & 2 \end{vmatrix} = (6, -2, -3).$$

So the triple vector product is

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -2 & 3 \\ 6 & -2 & -3 \end{vmatrix} = (12, 24, 8).$$

> *Turning to the right hand side,* $\boldsymbol{a}.\boldsymbol{c} = 12$ *and* $\boldsymbol{a}.\boldsymbol{b} = 20$*, so*
>
> $$\boldsymbol{b}(\boldsymbol{a}.\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}.\boldsymbol{b}) = 12(1,-3,4) - 20(0,-3,2) = (12,24,8),$$
>
> *as required.*

1.7 A Bit about Partial Derivatives

Reading: Section 3.1 Introduction (to vector calculus) (pp 437-444)

We will be using partial differentiation a lot in this course, but not in any very complicated way.

Suppose we have a function of more than one variable, $f(x, y)$. The the gradient of f going along a line of constant y is the partial derivative

$$\left.\frac{\partial f}{\partial x}\right|_y = \frac{\partial f}{\partial x}$$

The first form with the subscript is not normally used if the quantities being kept constant are just the other variables in the function - it is implied. This will always be the case here, and I will use the second form without a subscript.

The main result for partial derivatives that we will use is the *chain rule*. Suppose x and y are both functions of another variable, s. Then, the chain rule says that

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}. \tag{1.10}$$

If x and y are functions of two variables, say s and t, this generalises to

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}. \end{aligned} \tag{1.11}$$

Note that on the right hand side we are thinking of f as a function of the variables t and s, so the notation $\partial f/\partial s$ implies that the quantity kept constant is t.

We shall also use the identity

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},$$

which is valid for well behaved f.

Sometimes we shall be differentiating components of a vector which is itself a function of x, y, z, for example $F_x(x, y, z)$. In this notation, the subscript x is just a label for the component; when we differentiate this with respect to x, the differentiation does not do anything to the x in the subscript.

Example

> Consider the function $f(x, y) = x^2 - y^2$. Suppose we make a transformation of variables to r and θ (polar coordinates) given by
>
> $$x = r\cos\theta \qquad y = r\sin\theta.$$
>
> Use the chain rule formulae (with s and t replaced by r and θ) to work out $\partial f/\partial r$ and $\partial f/\partial \theta$. Express your answer in terms of the r and θ variables.
>
> By substituting for x and y, obtain an explicit expression for $f(r, \theta)$. Now use this expression to to work out $\partial f/\partial r$ and $\partial f/\partial \theta$. Verify that you get the same answer by this method.

For our f,

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = -2y$$

while

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

So

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = 2x\cos\theta - 2y\sin\theta = 2r(\cos^2\theta - \sin^2\theta).$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -2xr\sin\theta - 2yr\cos\theta = -4r^2\sin\theta\cos\theta.$$

Making the substitution

$$f(r,\theta) = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta).$$

So

$$\frac{\partial f}{\partial r} = 2r(\cos^2\theta - \sin^2\theta).$$

and

$$\frac{\partial f}{\partial \theta} = r^2(-2\cos\theta\sin\theta - 2\sin\theta\cos\theta = -4r^2\cos\theta\sin\theta,$$

as before.

2 Differentiation of Vectors

Suppose a vector, $\boldsymbol{a}$, is a function of a variable t. We can define the derivative of $\boldsymbol{a}$ with respect to t by

$$\frac{d\boldsymbol{a}}{dt} = \lim_{\delta t\to 0}\frac{\boldsymbol{a}(t+\delta t) - \boldsymbol{a}(t)}{\delta t}. \tag{2.1}$$

Note $d\boldsymbol{a}/dt$ is the difference of two vectors, so it is itself a vector.

If we have expressed the vector in terms of components, and the basis vectors are not themselves functions of t, the differentiation is straight forward. If

$$\boldsymbol{a}(t) = a_x(t)\boldsymbol{i} + a_y(t)\boldsymbol{j} + a_z(t)\boldsymbol{k} = (a_x(t),\, a_y(t),\, a_z(t)), \tag{2.2}$$

then

$$\frac{d\boldsymbol{a}}{dt} = \frac{da_x}{dt}\boldsymbol{i} + \frac{da_y}{dt}\boldsymbol{j} + \frac{da_z}{dt}\boldsymbol{k} = \left(\frac{da_x}{dt},\, \frac{da_y}{dt},\, \frac{da_z}{dt}\right). \tag{2.3}$$

2.1 Differentiating products

There are various products involving vectors we may wish to differentiate. All follow easily from the standard product rule of differentiation.

If $\phi(t)$ is a scalar function then

$$\frac{d}{dt}(\phi \boldsymbol{a}) = \frac{d\phi}{dt}\boldsymbol{a} + \phi\frac{d\boldsymbol{a}}{dt}. \tag{2.4}$$

To differentiate a dot product,

$$\frac{d}{dt}(\boldsymbol{a}.\boldsymbol{b}) = \frac{d\boldsymbol{a}}{dt}.\boldsymbol{b} + \boldsymbol{a}.\frac{d\boldsymbol{b}}{dt}. \tag{2.5}$$

For a cross product

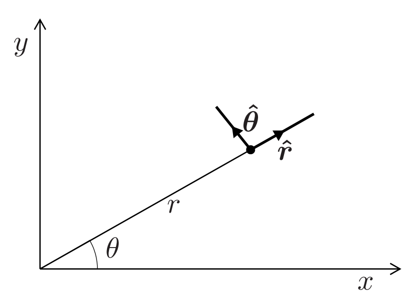
$$\frac{d}{dt}(\boldsymbol{a}\times\boldsymbol{b}) = \frac{d\boldsymbol{a}}{dt}\times\boldsymbol{b} + \boldsymbol{a}\times\frac{d\boldsymbol{b}}{dt}. \tag{2.6}$$

Example

> The angular momentum of a particle is given by
>
> $$\boldsymbol{L} = \boldsymbol{r}\times m\frac{d\boldsymbol{r}}{dt},$$
>
> where $\boldsymbol{r}(t)$ is the position at time t. Show that the rate of change of angular momentum is equal to the torque, that is
>
> $$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r}\times\boldsymbol{F}, \qquad \text{where} \qquad \boldsymbol{F} = m\frac{d^2\boldsymbol{r}}{dt^2}.$$

> *We have, assuming m is a constant,*
>
> $$\begin{aligned}\frac{d\boldsymbol{L}}{dt} &= \frac{d}{dt}\left(\boldsymbol{r}\times m\frac{d\boldsymbol{r}}{dt}\right) = m\left(\frac{d\boldsymbol{r}}{dt}\times\frac{d\boldsymbol{r}}{dt} + \boldsymbol{r}\times\frac{d^2\boldsymbol{r}}{dt^2}\right)\\ &= \boldsymbol{0} + \boldsymbol{r}\times m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{r}\times\boldsymbol{F}.\end{aligned}$$

2.1.1 Radial and Transverse Components of Velocity and Acceleration

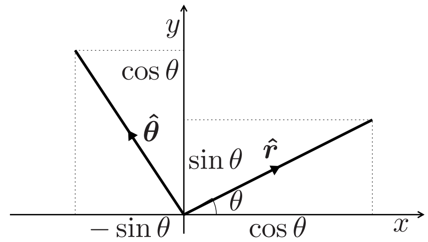


Sometimes, cartesian coordinates are not the best choice for solving a problem. For two dimensional problems, polar coordinates are sometimes more useful.

The polar (r, θ) coordinates are related to cartesian (x, y) coordinates by

$$\begin{aligned} x &= r\cos\theta\\ y &= r\sin\theta. \end{aligned} \tag{2.7}$$

We want to chose basis vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ such that $\hat{\boldsymbol{r}}$ points in the direction of increasing r, but constant θ, while $\hat{\boldsymbol{\theta}}$ points in the direction of increasing θ with r fixed. Thus $\hat{\boldsymbol{r}}$ is in the radial direction and $\hat{\boldsymbol{\theta}}$ is in the transverse direction.



We can see that

$$\begin{aligned}\hat{\boldsymbol{r}} &= \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta \\ \hat{\boldsymbol{\theta}} &= -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta.\end{aligned} \tag{2.8}$$

Note that $\hat{\boldsymbol{r}}.\hat{\boldsymbol{\theta}} = 0$; the basis vectors are *orthogonal*.

If the vector we are interested in is changing with time, the directions of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ will also be changing. We need to take this into account when we differentiate the vector to work out velocities and accelerations. Differentiating Eq.(2.8) we find

$$\begin{aligned}\frac{d\hat{\boldsymbol{r}}}{dt} &= \boldsymbol{i}\left(-\sin\theta\frac{d\theta}{dt}\right) + \boldsymbol{j}\left(\cos\theta\frac{d\theta}{dt}\right) = (-\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta)\frac{d\theta}{dt} \\ &= \hat{\boldsymbol{\theta}}\frac{d\theta}{dt} = \hat{\boldsymbol{\theta}}\,\dot{\theta}.\end{aligned} \tag{2.9}$$

and

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\boldsymbol{i}\left(\cos\theta\frac{d\theta}{dt}\right) + \boldsymbol{j}\left(-\sin\theta\frac{d\theta}{dt}\right) = -\hat{\boldsymbol{r}}\,\dot{\theta} \tag{2.10}$$

We can now use thse these to work out the components of the velocity. If $\boldsymbol{r}(t) = r(t)\hat{\boldsymbol{r}}$ we get the velocity

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}(t)}{dt} = \frac{d}{dt}(r(t)\hat{\boldsymbol{r}})$$

Hence, using the product rules,

$$\begin{aligned}\boldsymbol{v}(t) &= \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\boldsymbol{r}}{dt} \\ &= \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}.\end{aligned} \tag{2.11}$$

Differentiating again gives the acceleration

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}(\dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}})$$

Now

$$\frac{d}{dt}(\dot{r}\,\hat{\boldsymbol{r}}) = \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\frac{d\hat{\boldsymbol{r}}}{dt} = \ddot{r}\,\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}}$$

and

$$\frac{d}{dt}(r\dot{\theta}\,\hat{\boldsymbol{\theta}}) = \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\,\hat{\boldsymbol{r}}.$$

Putting these together gives the acceleration in polar coordinates

$$\begin{aligned}\boldsymbol{a} &= (\ddot{r}\,\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}}) + (\dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\,\hat{\boldsymbol{r}}) = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\,\hat{\boldsymbol{\theta}} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\boldsymbol{\theta}}\end{aligned} \tag{2.12}$$

If we also write the force acting on the particle in polar coordinates, $\boldsymbol{F} = F_r\,\hat{\boldsymbol{r}} + F_\theta\,\hat{\boldsymbol{\theta}}$, then the equations of motion for the radial and transverse components become

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$
$$F_\theta = \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) \qquad (2.13)$$

For a central force, that is, one directed towards the origin, $F_\theta = 0$ so $d(r^2\dot{\theta})/dt = 0$. This means that $r^2\dot{\theta}$ is a constant. In fact, multiplying by m gives the angular momentum $L = mr^2\dot{\theta}$, which is conserved for a central force.

Example

> Consider motion in a circle with radius R, for example, a bead moving on a wire loop. Show that the acceleration is given by
>
> $$\boldsymbol{a} = -\frac{v^2}{R}\,\hat{\boldsymbol{r}} + \frac{dv}{dt}\,\hat{\boldsymbol{\theta}},$$
>
> where v is the speed of the particle.

> *Since $r = R$, its derivatives must be zero: $\dot{r} = \ddot{r} = 0$. Then, using Eq.(2.11) for the velocity*
>
> $$\boldsymbol{v} = R\dot{\theta}\,\hat{\boldsymbol{\theta}}.$$
>
> *Hence the speed, $v = R\dot{\theta}$ and $\dot{\theta} = v/R$. Then, using Eq.(2.12) for the acceleration gives*
>
> $$\boldsymbol{a} = -R\dot{\theta}^2\,\hat{\boldsymbol{r}} + R\ddot{\theta}\,\hat{\boldsymbol{\theta}}$$
> $$= -\frac{v^2}{R}\,\hat{\boldsymbol{r}} + \frac{dv}{dt}\,\hat{\boldsymbol{\theta}}.$$

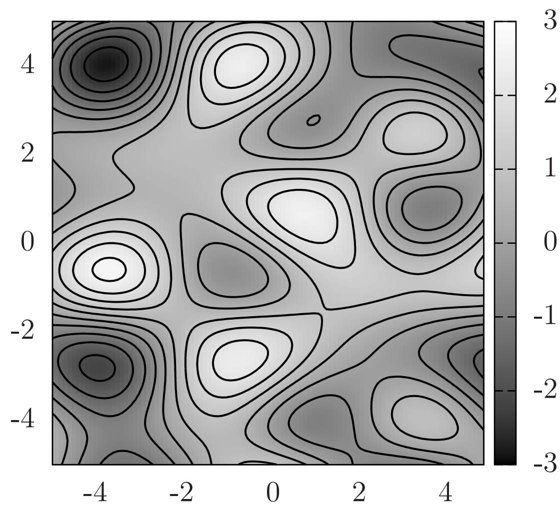
3 Scalar and Vector Fields

Reading: Section 3.1 Introduction (to vector calculus) (pp 434-437)

So far we have been considering dealing with just a single vector. We are now going to start working with scalar and vector *fields*. In a scalar field, we have a scalar quantity defined at every point in space, so it will be given by a function $\phi(x, y, z)$. Similarly, a vector field consists of a vector quantity defined at each point, $\boldsymbol{F}(x, y, z)$. Of course, the space can be one or two dimensional as well, but we will mainly be concerned with three dimensions here.

Scalar and vector fields should be very familiar concepts from physics. Examples of scalar fields include the electrostatic potential, $\phi(x, y, z)$, the temperature in a bar $T(x)$ and the height of a surface $h(x, y)$. Vector fields include the electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, the velocity in a moving fluid $\boldsymbol{v}(x, y, z)$ and the strain in a material under stress.

3.1 Visualising a Scalar Field



The two-dimensional scalar field $\phi(x, y) = \sin(1.3x)\cos(0.9y) + \cos(0.8x)\sin(1.9y) + \cos(0.2xy)$ plotted as contours and a greyscale.

A scalar field is often visualised by plotting *isosurfaces*, surfaces over which the field has a constant value. In three dimensions, these are two dimensional surfaces, while in two dimensions they are lines; the most familiar form is the contours on a map, which are a representation of the two dimensional scalar field corresponding to the height of the landscape. Of course, there are many other ways of visualising scalar fields, particularly in two dimensions, including grey scale and colour scale plots.

Formally, we can finding isosurfaces by solving the equation $\phi(x, y, z) = c$, where c is a constant which determines which isosurface we are dealing with. How easy this is to do will depend on the form of the function ϕ. We would normally chose to make the steps in c equal, so the spacing of the contours indicates how rapidly ϕ is changing. However, this is not compulsory or even always appropriate.

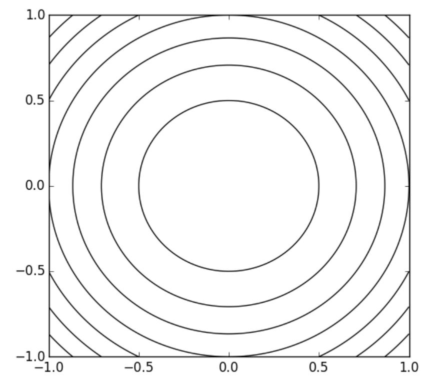
Example

Plot the isosurfaces (contours) of the two dimensional scalar field $\phi(x, y) = x^2 + y^2$

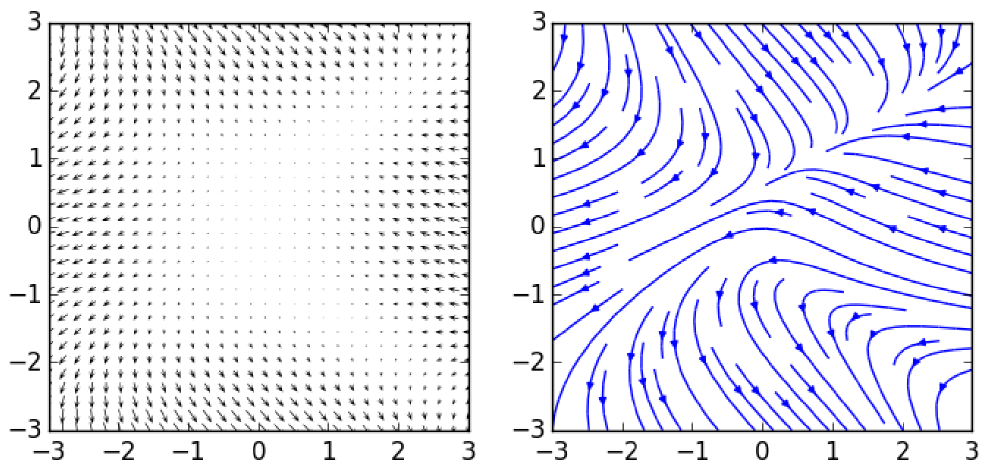
The contours are given by the equation

$$x^2 + y^2 = c,$$

which are circles of radius $r = \sqrt{c}$. Thus if we want to space the values of c equally, we need to take equal steps in r^2.



3.2 Visualising a Vector Field



The two-dimensional field $F_x = y^2 - x^2 - 1$, $F_y = x - y^2$ plotted as arrows and field lines

There is more than one way to picture a vector field. The most straight forward to construct is obtained by evaluating the field at a set of points on a grid and drawing a small arrow, with the corresponding magnitude and direction, at each point. This is sometimes called a *quiver plot*. However, that is not normally what we do in physics. Instead, we use field lines. A field line is a line, usually curved, such that the tangent at each point in space is in the direction of the field at that point. Note that this says nothing about the magnitude of the field. We can indicate this by how close together we plot the field lines: the higher the density, the stronger the field. However, there are many other ways, such as colour scales, line thicknesses etc. One of the reasons for using field lines is that they have a nice connection to the *divergence* and *curl* of the field, which we shall meet shortly.

3.2.1 Calculating Field Lines

In order to get more of an idea of what a field line means, suppose our vector field represents the velocity of particles in a moving fluid. Then, each field line represents the trajectory of a particle - the tangent to the trajectory points in the direction of the velocity. Hence we can calculate the field lines by integrating the equations of motion for the particle. If our velocity field is $\boldsymbol{F}(x, y, z)$, then the position of a point on the field line at time t, $\boldsymbol{r} = (x, y, z)$ is obtained by solving the differential equations

$$\frac{dx}{dt} = F_x(x, y, z) \qquad \frac{dy}{dt} = F_y(x, y, z) \qquad \frac{dz}{dt} = F_z(x, y, z). \tag{3.1}$$

This gives the parametric equation for a field line; as t is varied, our point $\boldsymbol{r}(t)$ traces out a field line. If $\boldsymbol{F}$ really is a velocity field (and it is independent of time), then t is a real time, so $\boldsymbol{r}(t)$ is actually the position of a particle at time t. However, the method works for any vector field, with t an arbitrary parameter if nothing is really moving.

Example

Consider a two dimensional field $\boldsymbol{F} = (-y, x)$. Obtain an equation for the field lines.

In this case, the differential equations, Eq.(3.1) are

$$\frac{dx}{dt} = F_x = -y$$
$$\frac{dy}{dt} = F_y = x$$

Differentiating the first of these again and substituting for dy/dt from the second, we find

$$\frac{d^2x}{dt^2} = -x.$$

This is the equation for a harmonic oscillator, so we know the general solution is

$$x = A\cos(t + \phi),$$

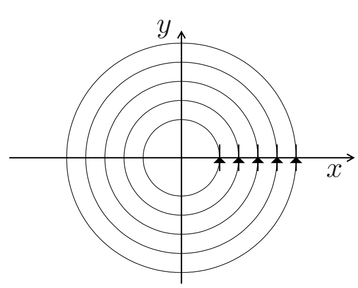
where A and ϕ are arbitrary constants. Then, differentiating,

$$y = -\frac{dx}{dt} = A\sin(t + \phi).$$

These represent the parametric equations for a field line - changing the constant A gives different field lines. In this case, it is easy to get rid of the parameter t by squaring and adding:

$$x^2 + y^2 = A^2[\cos^2(t + \phi) + \sin^2(t + \phi)] = A^2,$$

which is the equation for a circle of radius A.



Usually, it will not be possible to find analytic solutions by integrating Eq.(3.1) and we have to proceed numerically. This is a situation where it is very helpful to be able to use a computer package which can plot out field lines. An example is the python environment that you will be taught next year. The following python code produced the field line plot at the start of this section. Without needing to understand it in detail, you can change the definitions of F_x and F_y to plot other vector fields.

```
import numpy as np
import matplotlib.pyplot as plt

Y, X = np.mgrid[-3:3:100j, -3:3:100j]

Fx = -1 - X**2 + Y
Fy = 1 + X - Y**2

plt.figure()
plt.streamplot(X, Y, Fx, Fy, density=1)
plt.show()
```

4 Gradient of a Scalar Field

Reading: Section 3.2 Derivatives of a scalar point function (pp 451-455)

In cartesian coordinates, the *gradient* of a scalar field $\phi(x,y,z)$ is

$$\boldsymbol{\nabla}\phi = \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z}. \tag{4.1}$$

We normally say this as 'grad phi' or sometimes 'del phi'. $\boldsymbol{\nabla}\phi$ is a vector field. It can be considered as the result of the *vector operator*

$$\boldsymbol{\nabla} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{4.2}$$

acting on the field ϕ.

Example

> Consider the scalar field
>
> $$\phi(x,y,z) = x^3y + \cos(xy) + e^{2z}.$$
>
> Calculate the gradient, $\boldsymbol{\nabla}\phi$ and evaluate it at the point $(1, \pi/2, 0)$.

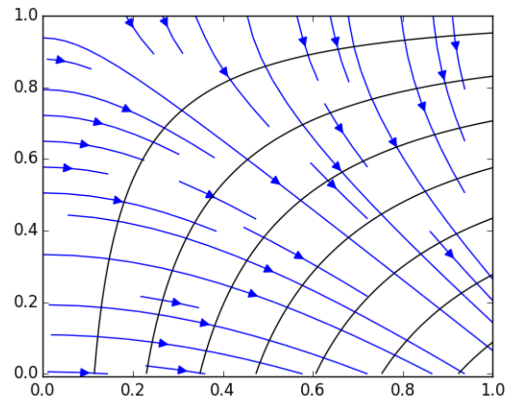
> *For this field, we have*
>
> $$\begin{aligned}\boldsymbol{\nabla}\phi &= \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z} = \boldsymbol{i}\,(3x^2y - y\sin(xy)) + \boldsymbol{j}\,(x^3 - x\sin(xy)) + \boldsymbol{k}\,2e^{2z}\\ &= \left(3x^2y - y\sin(xy),\ x^3 - x\sin(xy),\ 2e^{2z}\right)\end{aligned}$$
>
> *Evaluating this at the required point*
>
> $$\boldsymbol{\nabla}\phi(1,\pi/2,0) = (3\times1\times\pi/2 - \pi/2\times1,\ 1-1,\ 2) = (\pi, 0, 2).$$

4.1 Interpretation of $\boldsymbol{\nabla}\phi$

Consider the change in ϕ when moving from the point $\boldsymbol{r} = (x,y,z)$ to a nearby point $\boldsymbol{r} + \delta\boldsymbol{r} = (x+\delta x, y+\delta y, z+\delta z)$. By the rules of partial differentiation, this change is

$$\delta\phi \approx \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial z}\delta y + \frac{\partial\phi}{\partial z}\delta z = \boldsymbol{\nabla}\phi.\delta\boldsymbol{r}. \tag{4.3}$$

Now, suppose we take $\delta\boldsymbol{r}$ to be in the tangent plane of an isosurface, of ϕ. This means that, in the limit of $\delta r \to 0$, there will be no change in ϕ, that is $\delta\phi = 0$. Hence $\nabla\phi.\delta\boldsymbol{r} = 0$, which implies that $\nabla\phi$ is perpendicular to $\delta\boldsymbol{r}$ and so also to the tangent plane. So $\nabla\phi$ is a vector pointing in the direction of increasing ϕ.



Contours of a function $\phi = \sin(x)\cos(y + 1/2)$ plotted along with field lines (arrowed lines) of $\nabla\phi$.

Example

Given the surface $x^3y^2z = 12$, find the equation of the tangent plane at the point $\boldsymbol{p} = (1, -2, 3)$.

the surface is an isosurface of the function $\phi = x^3y^2z^2$. For this ϕ,

$$\nabla\phi = (3x^2y^2z,\ 2x^3yz,\ x^3y^2).$$

Evaluating this at the point $\boldsymbol{p}$ gives a vector, $\boldsymbol{n}$, normal to the tangent plane:

$$\boldsymbol{n} = (36, -12, 4).$$

Thus the equation of the plane must take the form

$$36x - 12y + 4z = constant.$$

Putting in the requirement for the plane to pass through $\boldsymbol{p}$ gives the value for the constant. Then (dividing through by 4) we get the equation of the required plane:

$$9(x - 1) - 3(y + 2) + (z - 3) = 0.$$

4.2 Directional Derivative

Suppose we want to find the rate of change of ϕ with distance, $d\phi/ds$, at a given point (x_0, y_0, z_0) along a given line. Let

$$\hat{\boldsymbol{u}} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$

be a *unit* vector directed along the line (so $a^2 + b^2 + c^2 = 1$).

A point on the line a distance s from (x_0, y_0, z_0) is

$$(x, y, z) = (x_0, y_0, z_0) + s\hat{\boldsymbol{u}}$$

or

$$
\begin{aligned}
x &= x_0 + as \\
y &= y_0 + bs \\
z &= z_0 + cs.
\end{aligned} \tag{4.4}
$$

These are the parametric equations for the line. If we substitute these expressions into $\phi(x, y, z)$, we get a function $\phi(s)$ of only one variable, s, the distance along the line measured from (x_0, y_0, z_0). Differentiating this function, using the rules of partial differentiation,

$$
\begin{aligned}
\frac{d\phi}{ds} &= \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} \\
&= \frac{\partial\phi}{\partial x}a + \frac{\partial\phi}{\partial y}b + \frac{\partial\phi}{\partial z}c = \nabla\phi.\hat{\boldsymbol{u}}.
\end{aligned} \tag{4.5}
$$

This quantity is called the *directional derivative* of ϕ along $\hat{\boldsymbol{u}}$.

Note that the directional derivative must have a value between between $-|\nabla\phi|$, if $\hat{\boldsymbol{u}}$ is in the opposite direction to $\nabla\phi$, and $+|\nabla\phi|$, if it is in the same direction.

Example

For the function

$$\phi = x^2 + y + 2xy + z^3 + 4,$$

find $\nabla\phi$ and the directional derivative of ϕ at the point $(1, -2, 1)$ in the direction $\boldsymbol{a} = (2, -1, 1)$.

We have

$$\nabla\phi = (2x + 2y, 1 + 2x, 3z^2) = (-2, 3, 3)$$

at the point indicated.

To find the directional derivative, we need a unit vector in the direction of $\boldsymbol{a}$*. This is*

$$\hat{\boldsymbol{u}} = \frac{a}{|\boldsymbol{a}|} = \frac{a}{\sqrt{2^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{6}}(2, -1, 1).$$

The directional derivative is then

$$\frac{d\phi}{ds} = \nabla\phi.\hat{\boldsymbol{u}} = \frac{1}{\sqrt{6}}(-2, 3, 3).(2, -1, 1) = -\frac{2\sqrt{6}}{3}.$$

5 Divergence and Curl of a Vector Field

Reading: Section 3.3 Derivatives of a vector point function (pp 455-462)

We have previously seen the vector operator

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

If we think of ∇ as a bit like an ordinary vector, we can consider the dot and cross product of it with another vector. To be meaningful, this other vector must be a vector field, so the differential operators have something to act on. It turns out these quantities are very important in physics. Note that ∇ is not really a vector - we cannot specify its direction or magnitude until it acts on some function. However, the notation is very suggestive, and we shall see that often we can treat ∇ like an ordinary vector.

The dot product of ∇ with a vector field $\boldsymbol{F}$ is called the *divergence* of $\boldsymbol{F}$ (usually said as 'div F'). In cartesian coordinates, the divergence of the field $\boldsymbol{F} = (F_x, F_y, F_z)$, is

$$\text{div}\,\boldsymbol{F} = \nabla.\boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{5.1}$$

Note that, as it is like a dot product of two vectors, $\nabla.\boldsymbol{F}$ is a *scalar field*.

Remember that each component of $\boldsymbol{F}$ is, in general, a function of all the coordinates, so $F_x = F_x(x, y, z)$ etc, and when we take the partial derivative $\partial F_x/\partial x$, we are keeping y and z constant.

The cross product of ∇ with $\boldsymbol{F}$ is called the *curl* of $\boldsymbol{F}$. In cartesian coordinates it is

$$\text{curl}\,\boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \boldsymbol{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right). \tag{5.2}$$

Since it is like a cross product of two vectors, $\nabla\times\boldsymbol{F}$ is a *vector field*.

It can be shown that knowledge of $\nabla.\boldsymbol{F}$ and $\nabla\times\boldsymbol{F}$, along with appropriate boundary conditions, is sufficient to define $\boldsymbol{F}$ uniquely. So we can think of the divergence and curl as the two gradients of a vector field.

We shall see expressions for divergence and curl in other coordinate systems in a later section.

Example

> Consider the vector field $\boldsymbol{F} = (x^2, xy, yz)$. Find $\nabla.\boldsymbol{F}$ and $\nabla\times\boldsymbol{F}$.

> *We have*
>
> $$\begin{aligned} \frac{\partial F_x}{\partial x} &= 2x, & \frac{\partial F_x}{\partial y} &= 0, & \frac{\partial F_x}{\partial z} &= 0 \\ \frac{\partial F_y}{\partial x} &= y, & \frac{\partial F_y}{\partial y} &= x, & \frac{\partial F_y}{\partial z} &= 0 \\ \frac{\partial F_z}{\partial x} &= 0, & \frac{\partial F_z}{\partial y} &= z, & \frac{\partial F_z}{\partial z} &= y. \end{aligned}$$
>
> *Then*
>
> $$\nabla.\boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 2x + x + y = 3x + y$$
>
> *and*
>
> $$\begin{aligned} \nabla\times\boldsymbol{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ &= (z - 0, 0 - 0, y - 0) = (z, 0, y). \end{aligned}$$

Example

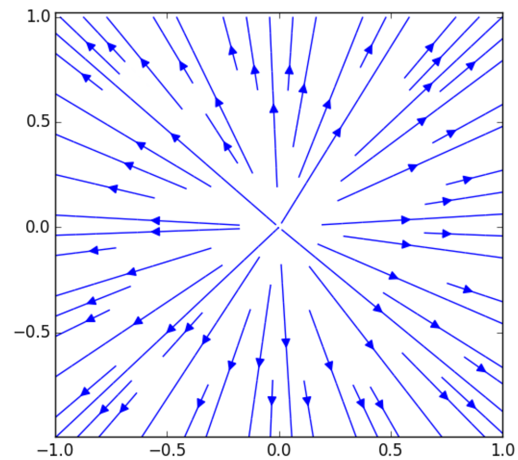
Show that $\nabla.\boldsymbol{r} = 3$ and $\nabla \times \boldsymbol{r} = \mathbf{0}$, where $\boldsymbol{r} = (x, y, z)$ is the position vector.

We get

$$\nabla.\boldsymbol{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

and

$$\nabla \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = (0, 0, 0) = \mathbf{0}.$$



*The vector field **r**, which represents a 'flow' outwards from the origin, has non-zero divergence everywhere, but zero curl.*

Example

Consider water flowing in a circular path, like stirring a cup of tea. A small volume of water at the point (x, y, z) at time t has coordinates $x = r\cos\omega t$, $y = r\sin\omega t$, $z = z_0$.

Calculate the velocity field

$$\boldsymbol{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

and determine $\nabla.\boldsymbol{v}$ and $\nabla \times \boldsymbol{v}$.

Differentiating, we have

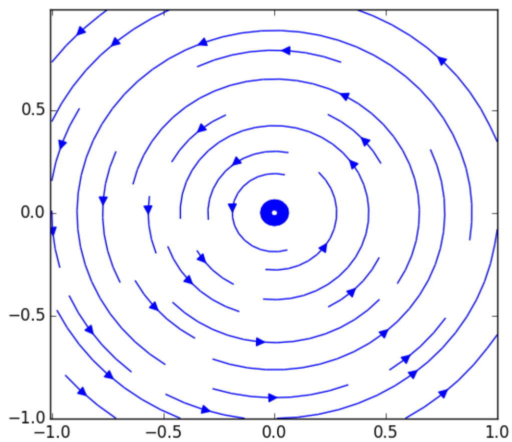
$$\boldsymbol{v} = (-r\omega\sin\omega t,\ r\omega\cos\omega t,\ 0) = \omega(-y, x, 0).$$

Then

$$\nabla.\boldsymbol{v} = \frac{\partial}{\partial x}(-\omega y) + \frac{\partial}{\partial y}(\omega x) + \frac{\partial}{\partial z}(0) = 0$$

and

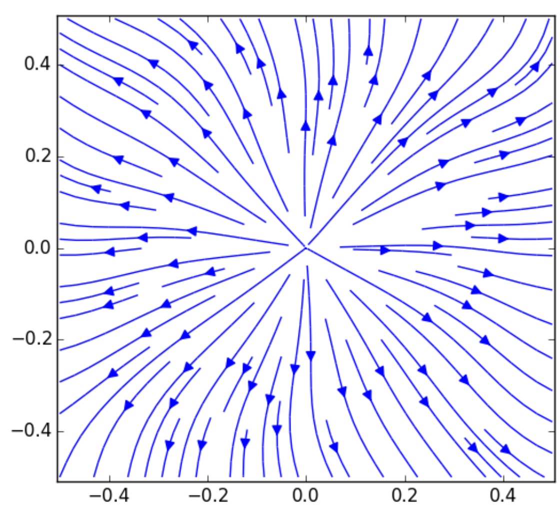
$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix} = \boldsymbol{k}\left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right) = 2\omega\boldsymbol{k}$$



This vector field, representing something rotating, has non-zero curl everywhere, but zero divergence

5.1 Physical Interpretation of Divergence and Curl

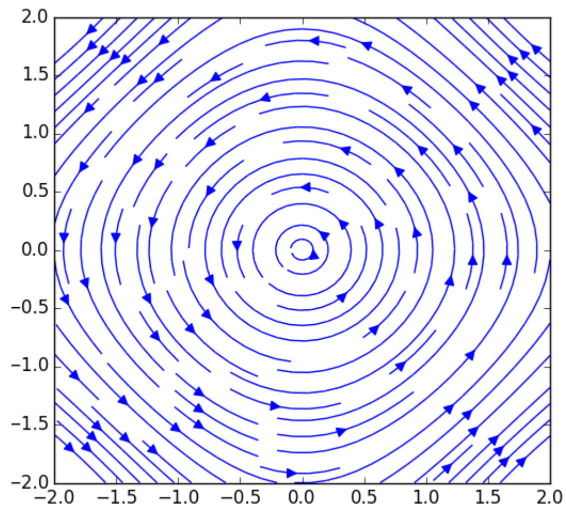
We saw in the previous examples that divergence is associated with outwards (or inwards) flow and curl with rotation. We shall now look at this a bit more carefully.



Field lines for the two dimensional vector function $\boldsymbol{F} = (x/(x^2+y^2)^{3/2} + \sin(10y), y/(x^2+y^2)^{3/2} + \cos(10x))$, which has $\nabla.\boldsymbol{F} \neq 0$ at the origin, but not elsewhere.

Suppose we have a field with zero divergence except at a single point in space. Then field lines will flow out from this point (or in towards it if the divergence is negative). If we think of our field lines as representing the flow of an incompressible fluid a region of non-zero divergence represents a source adding fluid to the system (if $\nabla.\boldsymbol{u} > 0$) or a sink removing it (if $\nabla.\boldsymbol{u} < 0$). You will later meet the *Divergence Theorem* which says that the total flow of fluid out of a region is equal to the integrated divergence of the field within the region.

If fluid is neither added or removed then the divergence must be zero everywhere.



Field lines for the two dimensional vector function $\boldsymbol{F} = (-\sin y, \sin x)$, which has zero divergence everywhere but finite curl. Note that streamplot has not made the lines continuous, even though $\nabla.\boldsymbol{F} = 0$ so they could be.

Now consider a field with zero divergence and non-zero curl. There is no need for the field lines to have ends, so they form loops. If, again, we think about a fluid then this means the fluid is rotating, with the direction of the rotation determined by the sign of the curl within the loop. If the curl is non-zero only at a single point the loops must all go around that point. If the divergence is non-zero, field lines can have ends and the possibilities are more complicated, but a non-zero curl is still associated with rotation. A nice way to think about it is to imagine swimming round a loop in the fluid; if it is easier to swim round one way than the other, the curl of the field inside the loop must be non-zero. You will meet *Stokes Theorem* which states that the circulation round a loop is equal to the integrated curl of the field within the loop.

5.2 Maxwell's Equations

Next year, you will meet Maxwell's equations which determine the electric field, $\boldsymbol{E}$, and magnetic field, $\boldsymbol{B}$. For the special case of no time dependence, and in free space, the equations are

$$\nabla.\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla.\boldsymbol{B} = 0 \tag{5.3}$$

$$\nabla \times \boldsymbol{E} = \boldsymbol{0} \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \tag{5.4}$$

Here, ρ is the charge density (a scalar) and $\boldsymbol{J}$ is the current density (a vector).

We see that, at least in the static case, for electric fields, there is zero curl and the divergence is non-zero only where there is charge. Electric field lines start and end on charges. For magnetic fields, there is always zero divergence, so field lines must form loops. Currents generate non-zero curl, so field lines loop around currents.

This term, in PHY102, you will find the fields due to various arrangements of charges and currents. In this section, we will show that a couple of these solutions do, as they must, satify Maxwell's equations.

Example

> The electric field from a point charge Q at the origin is,
>
> $$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{x}{r^3}\boldsymbol{i} + \frac{y}{r^3}\boldsymbol{j} + \frac{z}{r^3}\boldsymbol{k}\right),$$
>
> where $r^2 = x^2 + y^2 + z^2$. Show that $\nabla.\boldsymbol{E} = 0$ (except at the origin) and $\nabla \times \boldsymbol{E} = \boldsymbol{0}$.

For the given $\mathbf{E}$*, we have*

$$\nabla.\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right).$$

Now, using the quotient rule

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{(x^2+y^2+z^2)^{3/2} - x\frac{3}{2}(x^2+y^2+z^2)^{1/2}2x}{(x^2+y^2+z^2)^3} = \frac{1}{r^3} - \frac{3x^2}{r^5}.$$

Using the symmetry to get the other derivatives,

$$\nabla.\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5}\right) = 0,$$

as required.

Turning to the curl, consider first the $x-$*component,*

$$[\nabla\times\boldsymbol{E}]_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}.$$

For the given $\mathbf{E}$*,*

$$\frac{\partial E_z}{\partial y} = \frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{-3zy}{(x^2+y^2+z^2)^{5/2}}.$$

This is symmetric between x and z, so will be equal to $\partial E_y/\partial z$*. Hence the two terms cancel, giving* $[\nabla\times\boldsymbol{E}]_x = 0$*. By symmetry the other components are also zero, so* $\nabla\times\boldsymbol{E} = \mathbf{0}$ *as required.*

6 The Laplacian operator, ∇^2

If $\phi(x,y,z)$ is a scalar field then we know that

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

is a vector field. Taking the divergence of this vector field, we get

$$\nabla.(\nabla\boldsymbol{\phi}) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}, \tag{6.1}$$

which is, of course, a scalar. This combination of operators occurs often, so we abbreviate it, defining the *Laplacian* operator ∇^2 ('del-squared') by

$$\nabla^2\phi = \nabla.(\nabla\boldsymbol{\phi}) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{6.2}$$

The Laplacian operator can also act on a vector field $\boldsymbol{F} = (F_x, F_y, F_z)$. It then gives

$$\nabla^2\boldsymbol{F} = \frac{\partial^2\boldsymbol{F}}{\partial x^2} + \frac{\partial^2\boldsymbol{F}}{\partial y^2} + \frac{\partial^2\boldsymbol{F}}{\partial z^2} = \left(\nabla^2 F_x,\ \nabla^2 F_y,\ \nabla^2 F_z\right), \tag{6.3}$$

which is a vector field.

Example

The gravitational potential, ϕ, of a uniform spherical mass M centred on the origin is (for a point outside the mass)

$$\phi = \frac{GM}{r} = \frac{GM}{\sqrt{x^2 + y^2 + z^2}},$$

where G is Newton's gravitational constant.

Find $\nabla\phi$ in its simplest form and show that $\nabla^2\phi = 0$.

Using the chain rule,

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= GM\frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-1/2} = GM\left(-\tfrac{1}{2}\right)\left(x^2 + y^2 + z^2\right)^{-3/2}(2x) \\ &= -GM\frac{x}{(x^2 + y^2 + z^2)^{3/2}} = -GM\frac{x}{r^3}.\end{aligned}$$

Similarly, if we differentiate with respect to y or z, we get y/r^3 or z/r^3. Hence

$$\boldsymbol{\nabla}\phi = \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z} = -\frac{GM}{r^3}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = -\frac{GM}{r^3}\boldsymbol{r} = -\frac{GM}{r^2}\hat{\boldsymbol{r}},$$

which is the gravitational force on a unit mass due to the sphere.

Differentiating again using the product rule,

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2} &= -\frac{GM}{r^3} - GMx\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = -\frac{GM}{r^3} - GMx\left(-\tfrac{3}{2}\right)\frac{2x}{(x^2 + y^2 + z^2)^{5/2}} \\ &= -\frac{GM}{r^3} + GM\frac{3x^2}{r^5}.\end{aligned}$$

Doing the same for y and z, then adding, we get

$$\begin{aligned}\nabla^2\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -3\frac{GM}{r^3} + GM\frac{3(x^2 + y^2 + z^2)}{r^5} \\ &= -3\frac{GM}{r^3} + GM\frac{3r^2}{r^5} = 0.\end{aligned}$$

As we shall see later, it is much simpler to do this calculation using spherical polar coordinates.

7 Identities Involving Vector Operators

In this section we will look at the following identities, which apply for any scalar field ϕ and vector field $\boldsymbol{F}$,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = \boldsymbol{0} \tag{7.1}$$
$$\boldsymbol{\nabla}.(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0 \tag{7.2}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{F}) - \nabla^2\boldsymbol{F}. \tag{7.3}$$

7.1 Proofs

Since

$$\boldsymbol{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right),$$

taking the curl gives

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} = \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}, \frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}, \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \boldsymbol{0},$$

which proves identity (7.1).

Starting from

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right),$$

we get

$$\begin{aligned}\boldsymbol{\nabla}.(\boldsymbol{\nabla}\times\boldsymbol{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ &= \frac{\partial^2 F_z}{\partial x\partial y} - \frac{\partial^2 F_y}{\partial x\partial z} + \frac{\partial^2 F_x}{\partial y\partial z} - \frac{\partial^2 F_z}{\partial y\partial x} + \frac{\partial^2 F_y}{\partial z\partial x} - \frac{\partial^2 F_x}{\partial z\partial y} = 0,\end{aligned}$$

which is identity (7.2).

We could prove identity (7.3) in a similar way. It is a bit more messy, as we have to evaluate both sides and show they are equal. I shall leave this to be done as an exercise, and instead show you a useful trick which allows you to find such results much more quickly.

Exercise: Show, by direct differentiation, that identity (7.3) is correct.

7.2 Maxwell's Equations again

One of Maxwell's equations says that $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$. From Eq.(7.1), if we derive the electric field from a (scalar) potential, that is write $\boldsymbol{E} = \boldsymbol{\nabla}V$,

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}V) = \boldsymbol{0}.$$

Hence that Maxwell equation is automatically satisfied. To find V, we use

$$\boldsymbol{\nabla}.\boldsymbol{E} = \boldsymbol{\nabla}.(\boldsymbol{\nabla}V) = \nabla^2 V = \frac{\rho}{\varepsilon_0},$$

which is the Poisson equation.

We can do the same thing with the magnetic field, $\boldsymbol{B}$. In this case, we start from a *vector potential*, $\boldsymbol{A}$, with

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

Then, using Eq.(7.2), we get

$$\nabla.\boldsymbol{B} = \nabla.(\nabla \times \boldsymbol{A}) = 0,$$

which is one of Maxwell's equations. To find $\boldsymbol{A}$, we use

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla.\boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}.$$

If $\nabla.\boldsymbol{A} = 0$, and it turns out we can always chose this to be the case, $\boldsymbol{A}$ satisfies a vector version of the Poisson equation.

7.3 A Useful Trick

If we think of ∇ as like an ordinary vector, we can use results we already know about vectors to obtain identities like these. Note that the methods in this section do not provide formal proofs.

To obtain identity (7.3), recall the formula

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}.\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}.\boldsymbol{b}).$$

If we substitute $\boldsymbol{a} = \boldsymbol{b} = \nabla$ and $\boldsymbol{c} = \boldsymbol{F}$ we get the left hand side looking like (7.3). However, the right hand side does not make much sense, as the operators are the wrong side of the function. We have to chose the form we start from a bit more carefully. If, instead, we write

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}.\boldsymbol{c}) - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}, \tag{7.4}$$

which is equally correct for normal vectors, then do the same substitution, we get

$$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla.\boldsymbol{F}) - (\nabla.\nabla)\boldsymbol{F}. \tag{7.5}$$

Recalling that $\nabla.\nabla$ is the Laplacian operator ∇^2, we have 'obtained' identity (7.3).

The other two identities can be obtained in a similar way. Starting from $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$ we get $\nabla \times (\nabla\phi) = \boldsymbol{0}$. Using the identity

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c}.(\boldsymbol{a} \times \boldsymbol{b})$$

with $\boldsymbol{a} = \boldsymbol{b}$ gives

$$\boldsymbol{a}.(\boldsymbol{a} \times \boldsymbol{c}) = \boldsymbol{c}.(\boldsymbol{a} \times \boldsymbol{a}) = \boldsymbol{0}.$$

Then substituting $\boldsymbol{a} = \nabla$ and $\boldsymbol{c} = \boldsymbol{F}$ we get $\nabla.(\nabla \times \boldsymbol{F}) = 0$.

We can also generate new results. Suppose we want to work out $\nabla.(\boldsymbol{F} \times \boldsymbol{G})$, where $\boldsymbol{F}$ and $\boldsymbol{G}$ are both vector fields. Now, if this were just ordinary differerentiation, we would be using the product rule, something like

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}.$$

Another way to get this is to define which function the differential operator acts on by adding a superscript. Then, rather messily, we need to do

$$\frac{d}{dx}(fg) = \left(\frac{d^{(f)}}{dx} + \frac{d^{(g)}}{dx}\right) fg = \frac{d^{(f)}}{dx} fg + \frac{d^{(g)}}{dx} fg = g\frac{d^{(f)}}{dx} f + f\frac{d^{(g)}}{dx} g.$$

If we use the same notation for the vector calculus problem, we get

$$\nabla.(\boldsymbol{F} \times \boldsymbol{G}) = (\nabla^{(F)} + \nabla^{(G)}).(\boldsymbol{F} \times \boldsymbol{G}) = \nabla^{(F)}.(\boldsymbol{F} \times \boldsymbol{G}) + \nabla^{(G)}.(\boldsymbol{F} \times \boldsymbol{G}) \tag{7.6}$$

Next we need to use vector identities to get the operators in front of the function they act on. For the first term, using

$$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{b}),$$

we get

$$\boldsymbol{\nabla}^{(F)}.(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}.(\boldsymbol{\nabla}^{(F)}\times\boldsymbol{F}) = \boldsymbol{G}.(\boldsymbol{\nabla}\times\boldsymbol{F}).$$

Then, for the second term, we use

$$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}.(\boldsymbol{c}\times\boldsymbol{a}) = -\boldsymbol{b}.(\boldsymbol{a}\times\boldsymbol{c})$$

to get

$$\boldsymbol{\nabla}^{(G)}.(\boldsymbol{F}\times\boldsymbol{G}) = -\boldsymbol{F}.(\boldsymbol{\nabla}^{(G)}\times\boldsymbol{G}) = -\boldsymbol{F}.(\boldsymbol{\nabla}\times\boldsymbol{G}).$$

Putting these together gives

$$\boldsymbol{\nabla}.(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}.(\boldsymbol{\nabla}\times\boldsymbol{F}) - \boldsymbol{F}.(\boldsymbol{\nabla}\times\boldsymbol{G}). \tag{7.7}$$

This is correct, as can be proved by direct differentiation.

Exercise: Use these methods to 'show' that $\boldsymbol{\nabla}\times(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{F}(\boldsymbol{\nabla}.\boldsymbol{G}) - \boldsymbol{G}(\boldsymbol{\nabla}.\boldsymbol{F}) + (\boldsymbol{G}.\boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{F}.\boldsymbol{\nabla})\boldsymbol{G}$.

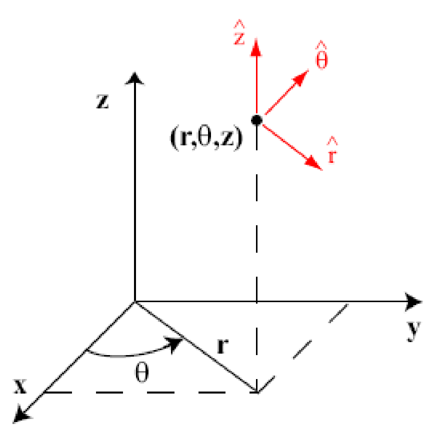
8 Vector Operators in other Coordinate Systems

Frequently in physics we will wish to use coordinate systems other than cartesians. This is usually because a problem will be easier to solve if our coordinates have the same symmetry as the physical system we are dealing with. For example, when dealing with systems with spherical symmetry, it is best to use spherical polar coordinates.

In this section, I shall go through the maths of transforming the gradient operator, $\boldsymbol{\nabla}V$, and and the divergence $\boldsymbol{\nabla}.\boldsymbol{F}$ into cylindrical polar coordinates. I shall also give results, but not proofs, for spherical polar coordinates.

Note, here I shall use V instead of ϕ as my generic scalar field, to avoid confusion with the ϕ in spherical polar coordinates.

8.1 Cylindrical Polar Coordinates



Cylindrical polar coordinates are a straightforward extension of the plane polar coordinates defined in Eq.(2.7) of Section 2.1:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \tag{8.1}$$

The unit vectors are in the directions given by increasing one coordinate while keeping the other two constant. As we have previously seen (Eq.(2.8)) they are given in terms of the cartesian unit vectors by

$$\begin{aligned} \hat{\boldsymbol{r}} &= \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta \\ \hat{\boldsymbol{\theta}} &= -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta \\ \hat{\boldsymbol{z}} &= \boldsymbol{k}. \end{aligned} \tag{8.2}$$

8.1.1 The gradient, ∇V

In cartesians, the gradient operator is

$$\nabla V = \frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k},$$

so to change to polars we need to convert both the differentials and the unit vectors.

First, from Eq.(8.1), using partial differentiation, we get

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}, \end{aligned} \tag{8.3}$$

with $\partial/\partial z$ unchanged.

Inverting these relationships gives

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}, \end{aligned} \tag{8.4}$$

while inverting the relationship for the unit vectors, Eq.(8.2), gives

$$\begin{aligned} \boldsymbol{i} &= \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \\ \boldsymbol{j} &= \hat{\boldsymbol{r}}\sin\theta - \hat{\boldsymbol{\theta}}\cos\theta. \end{aligned} \tag{8.5}$$

We can now work out

$$\begin{aligned} \boldsymbol{i}\frac{\partial V}{\partial x} &= (\hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\left(\cos\theta\frac{\partial V}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial V}{\partial \theta}\right) \\ &= \hat{\boldsymbol{r}}\left(\cos^2\theta\frac{\partial V}{\partial r} - \frac{1}{r}\cos\theta\sin\theta\frac{\partial V}{\partial \theta}\right) + \hat{\boldsymbol{\theta}}\left(-\sin\theta\cos\theta\frac{\partial V}{\partial r} + \frac{1}{r}\sin^2\theta\frac{\partial V}{\partial \theta}\right) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{j}\frac{\partial V}{\partial y} &= (\hat{\boldsymbol{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta)\left(\sin\theta\frac{\partial V}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial V}{\partial \theta}\right) \\ &= \hat{\boldsymbol{r}}\left(\sin^2\theta\frac{\partial V}{\partial r} + \frac{1}{r}\sin\theta\cos\theta\frac{\partial V}{\partial \theta}\right) + \hat{\boldsymbol{\theta}}\left(+\cos\theta\sin\theta\frac{\partial V}{\partial r} + \frac{1}{r}\cos^2\theta\frac{\partial V}{\partial \theta}\right) \end{aligned}$$

Combining these, several terms cancel, leaving

$$\begin{aligned}\nabla V &= \hat{\boldsymbol{r}}\left(\cos^2\theta\frac{\partial V}{\partial r} + \sin^2\theta\frac{\partial V}{\partial r}\right) + \hat{\boldsymbol{\theta}}\left(\frac{1}{r}\sin^2\theta\frac{\partial V}{\partial \theta} + \frac{1}{r}\cos^2\theta\frac{\partial V}{\partial \theta}\right) + \hat{\boldsymbol{z}}\frac{\partial V}{\partial z}\\ &= \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial V}{\partial z}\hat{\boldsymbol{z}}. \qquad (8.6)\end{aligned}$$

8.1.2 The divergence, $\nabla.\boldsymbol{F}$

In cartesians, the divergence of $\boldsymbol{F}$ is given by

$$\nabla.\boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

so to change to polars we need to convert the differentials, as above, and also the components of the vector $\boldsymbol{F}$.

Fo do this, we use the expressions for the unit vectors, Eq.(8.2), to write

$$\begin{aligned}\boldsymbol{F} &= F_r\hat{\boldsymbol{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_z\hat{\boldsymbol{z}}\\ &= F_r(\boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta) + F_\theta(-\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta) + F_z\boldsymbol{k}\\ &= \boldsymbol{i}(F_r\cos\theta - F_\theta\sin\theta) + \boldsymbol{j}(F_r\sin\theta + F_\theta\cos\theta) + \boldsymbol{k}F_z.\end{aligned}$$

We can now read off the components in cartesians,

$$\begin{aligned}F_x &= F_r\cos\theta - F_\theta\sin\theta\\ F_y &= F_r\sin\theta + F_\theta\cos\theta, \qquad (8.7)\end{aligned}$$

with F_z unchanged.

Using these, and the expressions for the differentials, we get

$$\begin{aligned}\frac{\partial F_x}{\partial x} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)(\cos\theta F_r - \sin\theta F_\theta)\\ &= \cos^2\theta\frac{\partial F_r}{\partial r} - \cos\theta\sin\theta\frac{\partial F_\theta}{\partial\theta} + \frac{1}{r}\sin^2\theta F_r - \frac{1}{r}\sin\theta\cos\theta\frac{\partial F_r}{\partial\theta} + \frac{1}{r}\sin\theta\cos\theta F_\theta + \frac{1}{r}\sin^2\theta\frac{\partial F_\theta}{\partial\theta}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F_y}{\partial y} &= \left(\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)(\sin\theta F_r + \cos\theta F_\theta)\\ &= \sin^2\theta\frac{\partial F_r}{\partial r} + \sin\theta\cos\theta\frac{\partial F_\theta}{\partial\theta} + \frac{1}{r}\cos^2\theta F_r + \frac{1}{r}\cos\theta\sin\theta\frac{\partial F_r}{\partial\theta} - \frac{1}{r}\cos\theta\sin\theta F_\theta + \frac{1}{r}\cos^2\theta\frac{\partial F_\theta}{\partial\theta}.\end{aligned}$$

Combining these, lots of terms cancel, leaving

$$\begin{aligned}\nabla.\boldsymbol{F} &= \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}\\ &= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}. \qquad (8.8)\end{aligned}$$

8.1.3 $\nabla^2 V$ and $\nabla\times\boldsymbol{F}$

Once we know how to transform ∇V and $\nabla.\boldsymbol{F}$, obtaining $\nabla^2 V$ is trivial. It is

$$\begin{aligned}\nabla^2 V = \nabla.(\nabla V) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial V}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2}. \qquad (8.9)\end{aligned}$$

The calculation of $\nabla \times \boldsymbol{F}$ proceeds in the same way as above, but it is more complicated because we have to transform the components of the vector function, the differentials and the unit vectors all at the same time. Here, I shall simply quote the result:

$$\nabla \times \boldsymbol{F} = \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ F_r & rF_\theta & F_z \end{vmatrix}$$
$$= \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\hat{\boldsymbol{z}} \tag{8.10}$$

Example

In cylindrical polar coordinates, the magnetic field due to a current carrying wire is

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\theta}}.$$

Show that $\nabla.\boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = \boldsymbol{0}$.

$\boldsymbol{B}$ has only a $\hat{\boldsymbol{\theta}}$ component, $F_\theta = \mu_0 I/2\pi r$, so

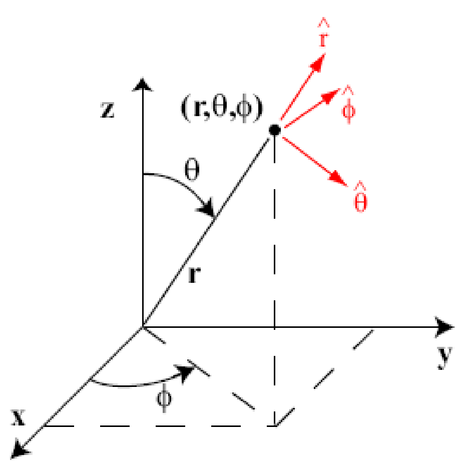
$$\nabla.\boldsymbol{B} = \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} = 0$$

and

$$\nabla \times \boldsymbol{B} = \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 0 & \mu_0 I/2\pi & 0 \end{vmatrix} = \boldsymbol{0},$$

because $\mu_0 I/2\pi$ is a constant.

8.2 Spherical Polar Coordinates



The other coordinate system which you will meet a lot in physics is spherical polars. For completeness, I shall give the expressions for the vector operators in spherical polars, but I shall not derive anything here.

In spherical polars

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned} \tag{8.11}$$

Note that spherical polar coordinates used by astronomers are not the same as this; in astronomy, the angle θ is measured from the equatorial plane, not the polar axis.

In spherical polar coordinates the gradient is

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{\phi}}. \tag{8.12}$$

The Laplacian is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}. \tag{8.13}$$

The divergence is

$$\boldsymbol{\nabla}.\boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}. \tag{8.14}$$

And the curl is

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{F} &= \begin{vmatrix} (1/r^2\sin\theta)\,\hat{\boldsymbol{r}} & (1/r\sin\theta)\,\hat{\boldsymbol{\theta}} & (1/r)\,\hat{\boldsymbol{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \\ &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}. \end{aligned} \tag{8.15}$$

Example

> Show, using spherical polar coordinates, that the gravitational potential,
>
> $$V(\boldsymbol{r}) = \frac{GM}{r},$$
>
> satisfies Laplace's equation $\nabla^2\phi = 0$.

> *Since V only depends on r, not θ and ϕ, the only non-zero term in the Laplacian is*
>
> $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{GM}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2\frac{GM}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\,(-GM) = 0$$

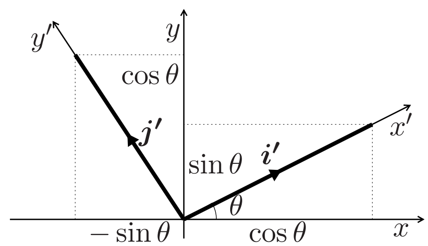
9 What are Scalars and Vectors?

Up until this point, I have used the term 'scalar' to apply to a simple number and 'vector' for something with magnitude and direction, or three numbers, corresponding to the components (in three dimensions). I shall now give a better definition, which depends on how the quantities involve change when we consider coordinate transformations. For *cartesian* scalars and vectors, the relevant transformation corresponds to a rotation of the axes.

Suppose you are given a quantity $\phi(x, y, z)$ in the x, y, z coordinate system and know that it becomes $\phi'(x', y', z')$ in the rotated x', y', z' system. If the quantity is unchanged in the primed coordinate system, that is $\phi' = \phi$, ϕ is said to be a (cartesian) scalar.

The definition of a vector is more complicated because the vector has a direction, so it will have different components when we rotate the axes to go between the primed and unprimed systems. We say that a set of three numbers forms a cartesian vector if they transform in the same way as a particular prototype vector, which is chosen to be the position vector $\boldsymbol{r} = (x, y, z)$.

The next thing to do is to see how a vector must transform. In these notes, I shall consider rotations only about the z-axis. For an arbitrary rotation axis, the maths becomes much more complicated, without actually providing anything new. We can always chose our coordinate system such that the z-axis is aligned with the direction of the rotation axis



Suppose we have an x, y, z coordinate system and define a new set of axes, x', y', z' rotated at an angle θ. From the figure, the unit vectors transform as

$$\begin{aligned} \boldsymbol{i}' &= \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta \\ \boldsymbol{j}' &= -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta \\ \boldsymbol{k}' &= \boldsymbol{k}. \end{aligned} \tag{9.1}$$

We can use this to see how to transform an arbitrary vector:

$$\begin{aligned} \boldsymbol{r} &= x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}' \\ &= x'(\boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta) + y'(-\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta) + z'\boldsymbol{k} \\ &= (x'\cos\theta - y'\sin\theta)\boldsymbol{i} + (x'\sin\theta + y'\cos\theta)\boldsymbol{j} + z'\boldsymbol{k}, \end{aligned}$$

from which we can read off the component transformation

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \\ z &= z'. \end{aligned} \tag{9.2}$$

So a quantity $\boldsymbol{a}$ is a cartesian vector if its components transform in this way, that is the quantitities a_x, a_y, a_z and a'_x, a'_y, a'_z are related by the same equations.

Note that the coordinate transformation equations can be written in matrix form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

As you saw in special relativity, coordinate transformations can be written using matrices.

So, how can you tell whether a quantity is a true scalar or vector? One way is that I can tell you; if I state that $\boldsymbol{a} = (1, 3, 2)$ is a vector, I am telling you not only what its value is in a particular coordinate system, but also how you have to transform it if you change to a different coordinate system. Another way is from physics; if the field $T(x, y)$ represents the temperature of a metal plate, we know that it must be a scalar field, because the temperature measured at a particular point cannot depend on the choice of coordinate system. Finally, we can prove that something is a scalar or vector, by working out how it transforms when we change the coordinate system. In the remainder of this section, I shall demonstrate this for some of the things I have called scalars and vectors elsewhere in these notes.

9.1 The Scalar Product

We can use these transformations to show that the scalar product is a true scalar, that is, the same in any coordinate system. Suppose we have two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. In the x, y, z coordinate system they have components (a_x, a_y, a_z) and (b_x, b_y, b_z). Using Eq.(9.2), we can write these in terms of the components in the x', y', z' system,

$$\begin{aligned} a_x &= a'_x \cos\theta - a'_y \sin\theta \qquad a_y = a'_x \sin\theta + a'_y \cos\theta \qquad a_z = a'_z \\ b_x &= b'_x \cos\theta - b'_y \sin\theta \qquad b_y = b'_x \sin\theta + b'_y \cos\theta \qquad b_z = b'_z. \end{aligned} \tag{9.3}$$

Hence the scalar product can be written as

$$\begin{aligned} \boldsymbol{a}.\boldsymbol{b} &= a_x b_x + a_y b_y + a_z b_z \\ &= (a'_x \cos\theta - a'_y \sin\theta)(b'_x \cos\theta - b'_y \sin\theta) + (a'_x \sin\theta + a'_y \cos\theta)(b'_x \sin\theta + b'_y \cos\theta) + a'_z b'_z \\ &= a'_x b'_x \cos^2\theta - a'_x b'_y \cos\theta \sin\theta - a'_y b'_x \sin\theta \cos\theta + a'_y b'_y \sin^2\theta \\ &\quad a'_x b'_x \sin^2\theta + a'_x b'_y \sin\theta \cos\theta + a'_y b'_x \cos\theta \sin\theta + a'_y b'_y \cos^2\theta + a'_z b'_z \\ &= a'_x b'_x (\cos^2\theta + \sin^2\theta) + a'_y b'_y (\sin^2\theta + \cos^2\theta) + a'_z b'_z \\ &= a'_x b'_x + a'_y b'_y + a'_z b'_z. \end{aligned}$$

We see that the scalar product comes out the same whichever coordinate system we work in - it is a cartesian scalar.

9.2 The Vector Product

In order to show that the vector product is a cartesian vector, we need to show that its components transform like Eq.(9.2). Using the results in Eq.(9.3), we find

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b})_x &= a_y b_z - a_z b_y \\ &= (a'_x \sin\theta + a'_y \cos\theta)\, b'_z - a'_z\, (b'_x \sin\theta + b'_y \cos\theta) \\ &= \sin\theta\, (a'_x b'_z - a'_z b'_x) + \cos\theta\, (a'_y b'_z - a'_z b'_y) = -\sin\theta\, (\boldsymbol{a} \times \boldsymbol{b})'_y + \cos\theta\, (\boldsymbol{a} \times \boldsymbol{b})'_x, \end{aligned}$$

as required. Similarly,

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b})_y &= a_z b_x - a_x b_z \\ &= a'_z\, (b'_x \cos\theta - b'_y \sin\theta) - (a'_x \cos\theta - a'_y \sin\theta)\, b'_z \\ &= \cos\theta\, (a'_z b'_x - a'_x b'_z) + \sin\theta\, (a'_y b'_z - a'_z b'_y) = \sin\theta\, (\boldsymbol{a} \times \boldsymbol{b})'_x + \cos\theta\, (\boldsymbol{a} \times \boldsymbol{b})'_y \end{aligned}$$

and

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b})_z &= a_x b_y - a_y b_x \\ &= (a'_x \cos\theta - a'_y \sin\theta)(b'_x \sin\theta + b'_y \cos\theta) - (a'_x \sin\theta + a'_y \cos\theta)(b'_x \cos\theta - b'_y \sin\theta) \\ &= a'_x b'_x \cos\theta \sin\theta + a'_x b'_y \cos^2\theta - a'_y b'_x \sin^2\theta - a'_y b'_y \cos\theta \sin\theta \\ &\quad - a'_x b'_x \cos\theta \sin\theta + a'_x b'_y \sin^2\theta - a'_y b'_x \cos^2\theta + a'_y b'_y \cos\theta \sin\theta \\ &= a'_x b'_y - a'_y b'_x = (\boldsymbol{a} \times \boldsymbol{b})'_z. \end{aligned}$$

Hence the vector product transforms like a cartesian vector when we rotate the axes.

9.3 Vector Operators

Using the methods of partial differentiation, we find

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} &= \frac{\partial\phi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial\phi}{\partial y'}\frac{\partial y'}{\partial x} = \cos\theta\frac{\partial\phi}{\partial x'} - \sin\theta\frac{\partial\phi}{\partial y'} \\
\frac{\partial\phi}{\partial y} &= \sin\theta\frac{\partial\phi}{\partial x'} + \cos\theta\frac{\partial\phi}{\partial y'} \\
\frac{\partial\phi}{\partial z} &= \frac{\partial\phi}{\partial z'}.
\end{aligned}
\tag{9.4}
$$

Comparing these with Eq.(9.2) we see that the components of $\boldsymbol{\nabla}\phi = (\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z)$ transform in exactly the same way as the components of $\boldsymbol{r} = (x, y, z)$, so they form a vector.

Next, let us show that $\boldsymbol{\nabla}.\boldsymbol{F}$ is, as I have claimed, a scalar. In the (x, y, z) coordinate system, it is

$$\boldsymbol{\nabla}.\boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

We now have to transform both the components of the vector (F_x, F_y, F_z), according to Eq.(9.2) and the operators $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ according to Eq.(9.4).

$$
\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\
&= \left(\cos\theta\frac{\partial}{\partial x'} - \sin\theta\frac{\partial}{\partial y'}\right)(\cos\theta F'_x - \sin\theta F'_y) + \left(\sin\theta\frac{\partial}{\partial x'} + \cos\theta\frac{\partial}{\partial y'}\right)(\sin\theta F'_x + \cos\theta F'_y) + \frac{\partial F'_z}{\partial z'} \\
&= \cos^2\theta\frac{\partial F'_x}{\partial x'} - \cos\theta\sin\theta\frac{\partial F'_y}{\partial x'} - \sin\theta\cos\theta\frac{\partial F'_x}{\partial y'} + \sin^2\theta\frac{\partial F'_y}{\partial y'} + \frac{\partial F'_z}{\partial z'} \\
&\quad + \sin^2\theta\frac{\partial F'_x}{\partial x'} + \sin\theta\cos\theta\frac{\partial F'_y}{\partial x'} + \cos\theta\sin\theta\frac{\partial F'_x}{\partial y'} + \cos^2\theta\frac{\partial F'_y}{\partial y'} + \frac{\partial F'_z}{\partial z'} \\
&= \frac{\partial F'_x}{\partial x'} + \frac{\partial F'_y}{\partial y'} + \frac{\partial F'_z}{\partial z'},
\end{aligned}
$$

as required.

Note that the maths we did here was exactly the same as we used to prove that the dot product is a scalar. We could also show that $\boldsymbol{\nabla}\times\boldsymbol{F}$ is a cartesian vector, but the proof is exactly the same as for the vector product, so I shall not do it here.

If we can write an equation in terms of vectors and scalars, we know it will be true for any such choice of axes. Since we would normally expect the properties of a physical system to be independent of our (arbitrary) choice of axes, the equations of physics tend to be written in this form. Note that the transformation properties under rotations define *cartesian* vectors and scalars. We can define other types of vector and scalars by their properties when they undergo different types of transformation. For example, in relativity, we define *four-vectors* (and scalars) by the way they change under Lorentz transformations. This means that if we can write an equation in terms of four-vectors and scalars, it will be valid in any (inertial) reference frame.