PHY120 - Unit 5: Differential equations

V. A. Kudryavtsev

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Web-page: http://kudryavtsev.staff.shef.ac.uk/phy120/ or link from 'list of modules'

1 Introduction, examples

The differential equation is an equation which includes one (or more) derivatives of one (or more) functions. It can also include the function itself and/or its argument(s). Examples of differential equations are:

$$\frac{dy}{dx} = x \tag{1}$$

$$\frac{dy}{dx} = y \cos x \tag{2}$$

$$\frac{d^2y}{dx^2} = \sin x \tag{3}$$

$$\frac{d^2I}{dt^2} = 2^t \tag{4}$$

The solution of the differential equation is a function (or all functions) which satisfies the equation. In the first three equations above the unknown functions are $y(x)$, in the Eq. (4) this function is $I(t)$.

The differential equations can be of different *orders* defined by the highest order derivative included. For instance, the Eqs. (1) and (2) are the 1st order equations whereas Eqs. (3) and (4) are the equations of the 2nd order.

Example:

Find the solution of the first order differential equation

$$\frac{dy}{dx} = x \tag{5}$$

The equation is the simplest one and the solution requires just the integration:

$$y = \int \frac{dy}{dx} dx = \int x dx = \frac{x^2}{2} \tag{6}$$

Let's now check whether the solution satisfies the initial equation. *It is very important to check the answer when solving any equation.*

$$\frac{dy}{dx} = \left(\frac{x^2}{2}\right)' = x \tag{7}$$

So the solution is correct but this is not the only solution. As in any integration without limits, called *indefinite integration*, the answer is valid to within a constant term c. In our case the function $y = x^2/2+c$ also satisfies the equation since the derivative of a constant is 0. So the general solution of the equation will be $y = x^2/2+c$. To avoid this uncertainty the so-called *initial condition* should be given. In our case the problem can be formulated in the following way. Find the solution of the first order differential equation

$$\frac{dy}{dx} = x \tag{8}$$

which satisfies also the (initial) condition: $y = 3$ when $x = 2$.

In this case we get: $3 = 2 + c$ and $c = 1$. So the solution is $y = x^2/2 + 1$.

See also Recommended textbook, Volume two, Chapter 20: Differential equations, Pages 742-756.

2 First order equations: special case of separable variables

In the example above the solution can be found by integrating the right-hand side of the equation. Similar examples are:

1.

$$\frac{dy}{dx} = x^2 - 3x - 1 \tag{9}$$

The solution is given by the integral:

$$y = \int \frac{dy}{dx}dx = \int (x^2 - 3x - 1)dx = \frac{x^3}{3} - \frac{3x^2}{2} - x + c \tag{10}$$

where c is a constant.

2.

$$\frac{dy}{dx} = \sin x \tag{11}$$

The solution is:

$$y = \int \frac{dy}{dx}dx = \int \sin x dx = -\cos x + c \tag{12}$$

where c is a constant.

3.

$$\frac{dy}{dx} = \frac{1}{x-1} \tag{13}$$

The solution is:

$$y = \int \frac{dy}{dx}dx = \int \frac{1}{x-1}dx = \ln|x-1| + c = \ln|c_1(x-1)| \tag{14}$$

where c and c_1 are constants and $c = \ln|c_1|$

In all equations above an initial condition (if given) can be used to find the constant c.

Generally, this type of equations can be written in the following form:

$$\frac{dy}{dx} = f(x) \tag{15}$$

The solution can be found by integrating the right-hand side of the equation.

$$y = \int \frac{dy}{dx}dx = \int f(x)dx \tag{16}$$

However, in a more general case the right-hand side of a differential equation can be presented as a function of x and y, y being itself a function of x.

$$\frac{dy}{dx} = f(x, y) \tag{17}$$

This type of equations is more difficult to solve and sometimes requires some special numerical methods of solving equations. We will consider some special cases for which the solution can be easily found analytically.

The simplest case is the differential equation of the form

$$\frac{dy}{dx} = f(x)g(y) \tag{18}$$

The solution can be found by *separating variables*: moving the function which depends on y to the left-hand side and leaving the function which depends on x (together with the differential dx) in the right-hand side of the equation.

$$\frac{dy}{g(y)} = f(x)dx \rightarrow \int \frac{dy}{g(y)} = \int f(x)dx \tag{19}$$

Then integrating both sides of the equation (and using the initial condition) the function $y(x)$ can be found. Let us consider a few examples.

Examples:

1. We considered already radioactive decay as an example of the exponential function, but we did not derive this formula. We can do this now using differential equation. In 1902 Rutherford and Soddy found that the rate of radioactive decay (the number of atoms of radioactive isotope decaying per second) is proportional to the number of atoms of this isotope present in a sample:

$$-\frac{dN}{dt} = kN \tag{20}$$

where k is the coefficient of proportionality. The minus sign is there because the number of radioactive atoms decreases with time – the derivative is negative (both sides of the equation should have the same sign – in this case they are positive). This is a differential equation which can be solved and the function $N(t)$ can be found. We need also an initial condition for normalization – the initial (at $t = 0$) number of radioactive atoms in the sample. Let this number be N_0.

$$-\frac{dN}{dt} = kN \to \int \frac{dN}{N} = -\int kdt \to \ln N = -kt + c \tag{21}$$

We use the initial condition that $N = N_0$ at $t = 0$. So $c = \ln N_0$.

$$\ln \frac{N}{N_0} = -kt \to N = N_0 e^{-kt} \tag{22}$$

The constant k is called the decay constant and the parameter $t_0 = 1/k$ is called the lifetime of the isotope. The equation can then be presented as

$$N = N_0 e^{-\frac{t}{t_0}} \tag{23}$$

The lifetime is connected to the half-life of the isotope, which is the time needed for a number of radioactive atoms to decrease to a half of the initial value.

$$\frac{N_0}{2} = N_0 e^{-\frac{t_{1/2}}{t_0}} \to \frac{1}{2} = e^{-\frac{t_{1/2}}{t_0}} \to \ln \frac{1}{2} = -\frac{t_{1/2}}{t_0} \to t_{1/2} = -t_0 \ln \frac{1}{2} \to t_{1/2} = t_0 \ln 2 \tag{24}$$

2. Another example from physics is the energy loss and propagation of charged subatomic particles through matter. A charged elementary particle is moving in matter interacting with atoms and losing its energy due to several processes. The equation which describes the energy loss is

$$-\frac{dE}{dx} = a + bE \tag{25}$$

where the parameters a and b depend on the material and the type of the subatomic particle but do not depend on E so are constant in this equation for certain particle type and material. The minus sign is there again because the energy of the particle decreases with the distance x (both sides of the equation should have the same sign – in this case they are positive). Let us assume that at the distance $x = 0$ the energy of the particle is $E = E_0$. The solution of the equation is

$$\begin{aligned} -\frac{dE}{dx} &= a + bE \to \int \frac{dE}{a + bE} = -\int dx \\ &\to \frac{1}{b}\left(\ln(a + bE)\right) = -x + c \end{aligned} \tag{26}$$

We use the initial condition of $E = E_0$ at $x = 0$. So $c = \frac{1}{b}\ln(a + bE_0)$.

$$\begin{aligned} \ln(a + bE) - \ln(a + bE_0) = -bx \quad &\to \quad \frac{a + bE}{a + bE_0} = e^{-bx} \\ &\to \quad a + bE = (a + bE_0)e^{-bx} \to E = \left(\frac{a}{b} + E_0\right) e^{-bx} - \frac{a}{b} \\ &\to \quad E = E_0 e^{-bx} - \frac{a}{b}\left(1 - e^{-bx}\right) \end{aligned} \tag{27}$$

3. Find a solution of the differential equation

$$\frac{dy}{dx} = \frac{xy^2 + x}{x^2 y - y} \tag{28}$$

with the initial condition $y = 1$ at $x = 0$.

$$\frac{dy}{dx} = \frac{xy^2 + x}{x^2y - y} = \frac{x(y^2+1)}{y(x^2-1)} \tag{29}$$

Separating variables

$$\frac{ydy}{y^2+1} = \frac{xdx}{x^2-1} \tag{30}$$

We can make a substitution: $y^2 + 1 = u$ and $x^2 - 1 = v$. Then $du = 2ydy$ or $dy = du/(2y)$; $dv = 2xdx$ or $dx = dv/(2x)$.

$$\frac{ydy}{y^2+1} = \frac{du}{2u}; \qquad \frac{xdx}{x^2-1} = \frac{dv}{2v} \tag{31}$$

$$\frac{du}{u} = \frac{dv}{v} \rightarrow \int \frac{du}{u} = \int \frac{dv}{v} \rightarrow \ln|u| = \ln|v| + c_1 \rightarrow \ln|u| = \ln|cv| \tag{32}$$

Taking an exponential from both sides

$$e^{\ln|u|} = e^{\ln|cv|} \rightarrow u = cv \tag{33}$$

Coming back to the original variables:

$$y^2 + 1 = c(x^2 - 1) \tag{34}$$

This gives the general solution:

$$y = \pm\sqrt{c(x^2-1) - 1} \tag{35}$$

From the initial condition we find that $c = -2$ and the final solution is:

$$y = \sqrt{1 - 2x^2} \tag{36}$$

Note that the particular solution which satisfies given initial conditions is valid for $-\sqrt{0.5} \leq x \leq \sqrt{0.5}$ and contains only positive part of the general solution.

[Check that the solution found, satisfies the original equation.]

You should be able to apply various methods of integration to solve differential equations.

Sometimes the differential equations are given in the form

$$(xy^2 + x)dx + (y - x^2y)dy = 0 \tag{37}$$

which can be re-arranged to obtain Eq. (28).

See also Recommended textbook, Volume two, Chapter 20: Differential equations, Pages 757-764.

3 First-order linear differential equations

Any differential equation which has a form

$$\frac{dy}{dx} + f(x)y = g(x) \tag{38}$$

is called first-order linear differential equation. First order – only first order derivative is present. Linear – only first degree terms in y' and y.

If $g(x) = 0$ then the equation can be solved by separating variables. In fact Eq. (38) can be re-arranged

$$\frac{dy}{y} = -f(x)dx \tag{39}$$

Some equations with $g(x) \neq 0$ can also be re-arranged to have separated variables. For example

$$\frac{dy}{dx} - xy = x \rightarrow \frac{dy}{dx} = x(y+1) \rightarrow \frac{dy}{y+1} = xdx \tag{40}$$

Now we will find out how to solve a first-order linear differential equation in a general form. Let us first multiply both sides of the Eq. (38) by a function $p(x)$ which should satisfy an additional condition:

$$p(x)\frac{dy}{dx} + p(x)f(x)y = \frac{d(yp(x))}{dx} \tag{41}$$

The original equation (38) thus becomes

$$p(x)\frac{dy}{dx} + p(x)f(x)y = p(x)g(x) \tag{42}$$

and the function $p(x)$ should satisfy Eq. (41). Note that Eqs. (41) and (42) have the same left-hand sides. Let us find this function $p(x)$ by solving Eq. (41). The right-hand side of Eq. (41) is

$$\frac{d(yp(x))}{dx} = p(x)\frac{dy}{dx} + \frac{dp(x)}{dx}y \tag{43}$$

So the Eq. (41) now becomes

$$p(x)\frac{dy}{dx} + p(x)f(x)y = p(x)\frac{dy}{dx} + \frac{dp(x)}{dx}y \tag{44}$$

$$\frac{dp(x)}{dx} = p(x)f(x) \tag{45}$$

We can separate variables p and x and get

$$\frac{dp}{p} = f(x)dx \tag{46}$$

Integrating

$$\ln|p| = \int f(x)dx \tag{47}$$

$$p(x) = e^{\int f(x)dx} \tag{48}$$

We can see that the function $p(x)$ given by Eq. (48) satisfies Eqs. (41) and (42) which have the same left-hand sides. So we can say that the original linear equation of the form

$$\frac{dy}{dx} + f(x)y = g(x) \tag{49}$$

has been reduced to the equation

$$\frac{d(yp(x))}{dx} = p(x)g(x) \tag{50}$$

obtained by combining Eqs. (41) and (42) where the function $p(x)$ is determined by Eq. (48) and is called the *integrating factor*. By integrating both sides of Eq. (50) we can obtain the solution for y.

$$yp(x) = \int p(x)g(x)dx \tag{51}$$

$$y = \frac{\int p(x)g(x)dx}{p(x)} \tag{52}$$

The integrating factor is calculated as an integral and hence an arbitrary constant should be added to it. This constant, however, will be cancelled anyway and so can be chosen from the point of view of convenience for further operations. In most cases it is convenient to set it equal to 0.

Example

Consider the differential equation

$$\frac{dy}{dx} + 2xy = 3x^2e^{-x^2} \tag{53}$$

This equation is of the first order and linear. The coefficient for the derivative term is 1. If it were not, all terms in the equation should be divided by this coefficient or function first. The equation has the form

$$\frac{dy}{dx} + f(x)y = g(x) \tag{54}$$

where the function $f(x) = 2x$. The integrating factor is then

$$p(x) = e^{\int 2xdx} = e^{x^2+c} = e^{x^2} \tag{55}$$

We take $c = 0$. Now we can multiply both sides of the original equation by this integrating factor to get

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 3x^2 \tag{56}$$

The left-hand side of this equation is equal to $\frac{d}{dx}\left(e^{x^2}y\right)$. So the original equation has been reduced to

$$\frac{d}{dx}\left(e^{x^2}y\right) = 3x^2 \tag{57}$$

This is similar to the general case Eq. (50). Integrating both sides of Eq. (57) we get

$$e^{x^2}y = \int 3x^2dx \tag{58}$$

$$y = \frac{\int 3x^2 dx}{e^{x^2}} \tag{59}$$

This solution is similar to the general case Eq. (52). This is easy to integrate and the final result is

$$y = (x^3 + c)e^{-x^2} \tag{60}$$

See also Recommended textbook, Volume two, Chapter 20: Differential equations, Pages 765-772.

4 First order equations: special case of $y = vx$ substitution

If a differential equation has a form of:

$$\frac{dy}{dx} = f(y/x) \tag{61}$$

(or can be re-arranged to have this form), then the solution can be found using a substitution $y = v(x)x$, where v is a function of x.

Making a substitution $y = v(x)x$, the left-hand side of the equation becomes

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \tag{62}$$

and the original equation is transformed to

$$x\frac{dv}{dx} + v = f(v) \rightarrow \frac{dv}{dx} = \frac{f(v) - v}{x} \rightarrow \frac{dv}{f(v) - v} = \frac{dx}{x} \tag{63}$$

So we have now an equation with separated variables.

Let us consider an example:

$$\frac{dy}{dx} = \frac{x + 3y}{2x} \tag{64}$$

The equation can be re-arranged to have $f(y/x)$

$$\frac{dy}{dx} = \frac{1 + 3\frac{y}{x}}{2} = f(y/x) \tag{65}$$

and we can use the substitution $y = v(x)x$.

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \rightarrow x\frac{dv}{dx} + v = \frac{x + 3vx}{2x} \rightarrow \frac{dv}{dx} = \frac{v + 1}{2x} \tag{66}$$

This is now an equation with separable variables.

$$\frac{dv}{v + 1} = \frac{dx}{2x} \rightarrow \ln|v + 1| = \frac{1}{2}\ln|x| + c_1 \rightarrow v + 1 = c\sqrt{x} \rightarrow v = c\sqrt{x} - 1 \tag{67}$$

Recalling now that $y = vx$ and $v = y/x$ we obtain

$$y = cx\sqrt{x} - x = c(x)^{3/2} - x \tag{68}$$

Checking the solution, left-hand side

$$\frac{dy}{dx} = \frac{3}{2}c\sqrt{x} - 1 \tag{69}$$

Right-hand side:

$$\frac{x + 3y}{2x} = \frac{x + 3cx\sqrt{x} - 3x}{2x} = \frac{3}{2}c\sqrt{x} - 1 \tag{70}$$

To summarize, we considered 3 types of the first-order differential equations according to the methods of their solution:

1. Equations with separable variables.

$$\frac{dy}{dx} = f(x)g(y) \rightarrow \int \frac{dy}{g(y)} = \int f(x)dx \tag{71}$$

2. Linear equations.

$$\frac{dy}{dx} + f(x)y = g(x) \rightarrow p(x) = e^{\int f(x)dx} \rightarrow \frac{d(yp(x))}{dx} = p(x)g(x) \rightarrow y = \frac{\int p(x)g(x)dx}{p(x)} \tag{72}$$

3. Special case of $y = vx$ substitution.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \rightarrow y = v(x)x \rightarrow x\frac{dv}{dx} = f(v) - v \tag{73}$$

Some equations can be attributed to two or more types. A linear equation can have separable variables and can be solved also using $y = vx$ substitution. For example $\frac{dy}{dx} + \frac{y}{x} = 0$.

You should be able to solve these types of equations. You should be able to identify the method and apply it to solve an equation. Do not forget to check the answer by substituting the solution in the original equation.

5 Second-order linear differential equations with constant coefficients

We will consider here some cases of second-order linear differential equations. The general form of such an equation is

$$\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + h(x)y = f(x) \tag{74}$$

It is difficult to solve such an equation and for most cases this cannot be done analytically. We will consider here special cases when such an equation can be solved analytically. We will look at the equations with constant coefficients: $g(x) = a$ and $h(x) = b$.

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \tag{75}$$

A few examples from physics:

1. Suppose there is an object falling down from the height h under the influence of gravity. Its velocity is determined by the first derivative of the function h: $v = h' = \dot{h} = \frac{dh}{dt}$ (we use $\dot{h}$ as a time derivative of a function h). The velocity (directed down) is antiparallel to the height (measured from the surface and directed up), so the velocity is negative here. The acceleration a is determined by the first derivative of the velocity or the second derivative of the height: $a = \dot{v} = \frac{dv}{dt} = \ddot{h} = \frac{d^2h}{dt^2}$. Newton's Second Law can then be written as (neglecting the air resistance):

$$m\frac{d^2h}{dt^2} = F \tag{76}$$

The force F is equal to the gravitational force: $F = -mg$, where m is the mass of the object and $g = 9.8$ m/s^2 is the acceleration due to gravity. The minus sign means that the force acts in the direction (down) antiparallel to that in which the height is measured (up). In other words, the object is accelerating in the direction opposite to the positive height axis, so $\ddot{h}$ is negative, whereas the absolute value of mg is positive. As both sides of the equation should have equal signs, a minus sign is assigned to the right-hand side of the equation. So Newton's Second Law becomes:

$$\frac{d^2h}{dt^2} = -g \tag{77}$$

This equation is the linear differential equation of the second order with coefficients $a = b = 0$. Obviously this is very easy to solve to find the height h as a function of time t.

In general, if the coefficient $b = 0$, then Eq. (75) can be solved by *reducing the order of the equation*. We can do this by replacing the first derivative of the function y with another function: $\frac{dy}{dx} = u(x)$ and the 2nd derivative with the first derivative of the new function: $\frac{d^2y}{dx^2} = \frac{du}{dx}$. Eq. (75) with $b = 0$ now becomes the 1st order differential equation:

$$\frac{du}{dx} + au = f(x) \tag{78}$$

which can be solved in many cases by separating variables or using integrating factor.

In our example above with an object falling down from a roof, both a and b are equal to zero. So we can reduce the order of the equation by defining a new function (velocity): $v = dh/dx$ and rewriting Eq. (77) as: $dv/dt = -g$. Now we can integrate both sides of the equation (or separate variables) to get $v = -gt + c$. If the initial velocity of the object is given ($v(t = 0) = v_0$), then we can find the constant c by substituting $t = 0$ and $v = v_0$ in the equation: $v_0 = -0 + c$, $c = v_0$ and $v = -gt + v_0$.

Recalling that $v = dh/dx$, we get now an equation for dh/dx: $v = dh/dt = -gt + v_0$. Integrating both sides again (or separating variables) we obtain: $h = -gt^2/2 + v_0t + c_1$. If an initial height of the object is given ($h = h_0$), then the constant c_1 can be found by substituting $t = 0$ and $h = h_0$ in the equation above: $h_0 = -0+0+c_1$ leading to $c_1 = h_0$ and the final answer: $h = -gt^2/2+v_0t+h_0$. If, for instance the initial velocity $v(0) = v_0 = 0$ and initial height $h(0) = h_0 = 20$ m then the solution becomes: $h = -gt^2/2 + 20$, where h is in metres.

So far we always neglected the air resistance. Now we can introduce the air resistance. This is similar to the viscosity of the medium in which case the force is proportional to the velocity of the object: $F_{res} = k|v| = -k\frac{dh}{dt}$. This force acts in the direction antiparallel to the gravitational force and antiparallel to the velocity, but parallel to the positive height axis. So the equation describing Newton's Second Law becomes:

$$m\frac{d^2h}{dt^2} = -mg + F_{res} = -mg + k|v| = -mg - k\frac{dh}{dt} \tag{79}$$

where the last term in the right-hand side of the equation describes the air resistance in the direction antiparallel to the gravitational force (h' is negative). Re-arranging this equation we get:

$$m\frac{d^2h}{dt^2} + k\frac{dh}{dt} = -mg \tag{80}$$

or

$$\frac{d^2h}{dt^2} + \frac{k}{m}\frac{dh}{dt} = -g \tag{81}$$

Eq. (81) is similar to Eq. (75) with $a = \frac{k}{m}$, $b = 0$ and $f(x) = -g$ (independent of x).

Eq. (80) can also be written as ($v = h'$, $a = v' = h''$)

$$m\frac{dv}{dt} + kv = -mg \tag{82}$$

This equation can be solved by separating variables giving the velocity as a function of time: $v = f(t)$. Then this equation $\frac{dh}{dt} = f(t)$ can also be solved by separating variables to give the height h as a function of time.

In general, Newton's Second Law can be written as:

$$m\frac{d^2x}{dt^2} = \sum F \tag{83}$$

where x is the distance of an object from an initial point, m is the mass of the object, $a = \frac{d^2x}{dt^2}$ is the acceleration of the object and $\sum F$ is the sum of all forces applied to the object.

2. Consider the motion of an object of mass m on a spring. If the object is moved from an equilibrium position by a force $F(t)$, then there are also two other forces acting on the object: the force caused by the stiffness of the spring – proportional to the distance from the equilibrium position, and the force caused by the viscosity of the medium (or the resistance of the air, or the frictional resistance) – proportional to the velocity of the object. Then Newton's Second Law describing the motion of the object – the distance x from an equilibrium position along the spring as a function of time t, can be written as:

$$m\frac{d^2x}{dt^2} = F(t) - K\frac{dx}{dt} - sx \tag{84}$$

where K is the coefficient of proportionality between the resistance force and the velocity of the object, s is the spring constant. The minus signs indicate that the forces are applied in the direction opposite to the external force and the motion of the object. The equation can be re-arranged:

$$\frac{d^2x}{dt^2} + \frac{K}{m}\frac{dx}{dt} + \frac{s}{m}x = \frac{F(t)}{m} \tag{85}$$

Obviously, Eq. (85) is similar to Eq. (75) with coefficients $a = \frac{K}{m}$ and $b = \frac{s}{m}$, and the function $f = \frac{F(t)}{m}$. This equation also describes the motion of a piston controlled by a spring in a cylinder with a frictional resistance of the cylinder walls.

3. Similar equation can be written for an electric circuit with an external voltage source $V(t)$, capacitor with a capacitance C, resistor R and inductance L:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t) \tag{86}$$

or

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{V(t)}{L} \tag{87}$$

Eq. (87) is similar to Eq. (75) with coefficients $a = \frac{R}{L}$ and $b = \frac{1}{LC}$, and the function $f = \frac{V(t)}{L}$.

The solution of the equations of this type is derived below. We start with the simplest case of $f(x) = 0$.

5.1 Second-order homogeneous linear differential equations

An equation of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{88}$$

is called *homogeneous or unforced.*

Let us take a guess what the solution can be. An obvious function to consider as a solution is an exponential. This is because the derivative of the exponential is equal to the original function. So we will look for a solution in the form $y = Ae^{mx}$. Substituting this solution into the original equation we get:

$$Am^2e^{mx} + aAme^{mx} + bAe^{mx} = 0 \tag{89}$$

Obviously $y = Ae^{mx} \neq 0$ so we can divide both sides of the equation by Ae^{mx}:

$$m^2 + am + b = 0 \tag{90}$$

This equation is called *auxiliary or characteristic equation.* The solution of the original Eq. (88) depends on the roots of the quadratic Eq. (90). We can consider 3 cases:

1. Auxiliary equation has 2 independent real roots. This happens if

$$a^2 - 4b > 0 \tag{91}$$

Then the solutions of the auxiliary equation are

$$m_{1,2} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right) \tag{92}$$

and the two solutions of the original Eq. (88) are

$$\begin{aligned} y_1 &= e^{m_1x} \\ y_2 &= e^{m_2x} \end{aligned} \tag{93}$$

The original Eq. (88) has a general solution as a linear combinations of the two solutions (Eq. 93)) determined by the roots of the auxiliary Eq. (92).

$$y = Ae^{m_1x} + Be^{m_2x} \tag{94}$$

where $m_{1,2}$ are the solutions of the auxiliary equation, and A and B are arbitrary constants which can be determined if two initial conditions are given. It is easy to prove that the general solution satisfies the initial equation. [Prove this].

If $y = u$ and $y = v$ are solutions of an equation, then the linear combination of these solutions $y = Au + Bv$, where A and B are arbitrary constants, is also a solution of the

equation. So in order to obtain the general solution of the 2nd order differential equation it is sufficient to obtain any two *linearly independent* solutions. (The two functions u and v are called *linearly independent* if neither u nor v is a constant multiple of the other (there is no constant c such that $u = cv$) or, in other words, the ratio u/v is not a constant but is a function of the argument (for instance x)).

We can prove the above theorem by substituting the linear combination $y = Au + Bv$ into the original Eq. (88) taking into account the u and v are the linearly independent solutions of this equation.

$$\frac{dy}{dx} = A\frac{du}{dx} + B\frac{dv}{dx} \tag{95}$$

$$\frac{d^2y}{dx^2} = A\frac{d^2u}{dx^2} + B\frac{d^2v}{dx^2} \tag{96}$$

Thus

$$\begin{aligned} \frac{d^2y}{dx^2} &+ a\frac{dy}{dx} + by \\ &= A\frac{d^2u}{dx^2} + B\frac{d^2v}{dx^2} + a\left(A\frac{du}{dx} + B\frac{dv}{dx}\right) + b(Au + Bv) \\ &= A\left(\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu\right) + B\left(\frac{d^2v}{dx^2} + a\frac{dv}{dx} + bv\right) \\ &= 0 + 0 = 0 \end{aligned} \tag{97}$$

So when solving the homogeneous Eq. (88), we looked for a solution in the form Ae^{mx}, found auxiliary equation and its two roots, determined the two solutions to Eq. (88) and obtained the general solution as a linear combination of these solutions.

2. Auxiliary equation has 1 root (two roots are equal). This happens if

$$a^2 - 4b = 0 \tag{98}$$

In this case the solution of the auxiliary equation is

$$m = -\frac{a}{2} \tag{99}$$

It turns out that the solution of the original equation is given by a linear combination of two functions: e^{mx} and xe^{mx}. So

$$y = Ae^{mx} + Bxe^{mx} = e^{mx}(A + Bx) \tag{100}$$

Checking this

$$\frac{dy}{dx} = Ame^{mx} + Bmxe^{mx} + Be^{mx} = e^{mx}(Am + B) + Bmxe^{mx} \tag{101}$$

$$\frac{d^2y}{dx^2} = me^{mx}(Am + B) + Bm^2xe^{mx} + Bme^{mx} = me^{mx}(Am + 2B) + Bm^2xe^{mx} \tag{102}$$

Substituting this in the original equation

$$me^{mx}(Am + 2B) + Bm^2xe^{mx} + ae^{mx}(Am + B) + aBmxe^{mx} + bAe^{mx} + bBxe^{mx} = 0 \tag{103}$$

Dividing by e^{mx} and re-arranging

$$A(m^2 + am + b) + Bx(m^2 + am + b) + B(2m + a) = 0 \tag{104}$$

As $m = -\frac{a}{2}$ and $m^2 + am + b = 0$ (the auxiliary equation) we get that the left-hand side of the original equation is equal to 0 meaning that the solution is correct.

3. Auxiliary equation does not have real roots, but has complex roots. This happens if

$$a^2 - 4b < 0 \tag{105}$$

Then the solutions of the auxiliary equation are:

$$m_{1,2} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right) = \frac{1}{2}\left(-a \pm \sqrt{-|a^2 - 4b|}\right) = \frac{1}{2}\left(-a \pm i\sqrt{|a^2 - 4b|}\right) \tag{106}$$

We can also write the solutions as

$$m_{1,2} = \alpha \pm i\omega \tag{107}$$

where $\alpha = -\frac{a}{2}$ and $\omega = \frac{\sqrt{|a^2-4b|}}{2}$.
Then the two basic solutions of the original Eq. (88) are

$$\begin{aligned} y_1 &= e^{m_1 x} = e^{(\alpha + i\omega)x} \\ y_2 &= e^{m_2 x} = e^{(\alpha - i\omega)x} \end{aligned} \tag{108}$$

From Unit 3 (Complex numbers) you should know that $e^{ix} = \cos x + i \sin x$. So

$$\begin{aligned} y_1 &= e^{(\alpha + i\omega)x} = e^{\alpha x}e^{i\omega x} = e^{\alpha x}(\cos \omega x + i \sin \omega x) \\ y_2 &= e^{(\alpha - i\omega)x} = e^{\alpha x}e^{-i\omega x} = e^{\alpha x}(\cos \omega x - i \sin \omega x) \end{aligned} \tag{109}$$

The two complex functions above form the basis for the general solution as a linear combination of the complex functions. We may want, however, the general solution written as a linear combination of two real functions. Recalling that if there are two linearly independent solutions of an equation, then its linear combination is also a solution of the equation, we can construct two linear combinations of the solutions given by Eq. (109) as

$$\begin{aligned} y_a &= \frac{1}{2}(y_1 + y_2) = e^{\alpha x}\cos \omega x \\ y_b &= -\frac{i}{2}(y_1 - y_2) = e^{\alpha x}\sin \omega x \end{aligned} \tag{110}$$

Then the general solution can be written as a linear combination of the two solutions given by Eq. (110)

$$y = y_a + y_b = e^{\alpha x}(A \cos \omega x + B \sin \omega x) \tag{111}$$

Another way to write down the general solution is

$$y = Ce^{\alpha x}\cos(\omega x + \phi) \tag{112}$$

where $C = \sqrt{A^2 + B^2}$ and $\phi = \arccos(A/C)$, $\phi = \arcsin(-B/C)$. This was shown by Eq. (24) in Unit 2.

In all cases considered above, the coefficients A and B can be determined if two initial conditions are given: $y(x_0) = y_0$ and $y'(x_1) = \frac{dy}{dx}(x_1) = y_1$.

To summarize, second-order linear homogeneous differential equations with constant coefficients of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{113}$$

can be solved using auxiliary equation

$$m^2 + am + b = 0 \tag{114}$$

The solutions of the original equations depend on the roots of the quadratic auxiliary equation:

1. If $a^2 - 4b > 0$, then the solutions of the auxiliary equation are real: $m_{1,2} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$ and the original homogeneous equation has the general solution:

$$y = Ae^{m_1 x} + Be^{m_2 x} \tag{115}$$

2. If $a^2 - 4b = 0$, then there is only one solution of the auxiliary equation: $m = -\frac{a}{2}$ and the original homogeneous equation has the general solution:

$$y = e^{mx}(A + Bx) \tag{116}$$

3. If $a^2 - 4b < 0$, then the solutions of the auxiliary equation are complex: $m_{1,2} = \alpha \pm i\omega$ where $\alpha = -\frac{a}{2}$ and $\omega = \frac{\sqrt{|a^2-4b|}}{2}$, and the original homogeneous equation has the general solution:

$$y = e^{\alpha x}(A\cos\omega x + B\sin\omega x) \tag{117}$$

or

$$y = Ce^{\alpha x}\cos(\omega x + \phi) \tag{118}$$

where $C = \sqrt{A^2 + B^2}$ and $\phi = \arccos(A/C)$, $\phi = \arcsin(-B/C)$.

See also Recommended textbook, Volume two, Chapter 20: Differential equations, Pages 783-795.

5.2 Second-order non-homogeneous linear differential equations

Second-order *non-homogeneous* or *inhomogeneous* linear differential equations with constant coefficients have the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \tag{119}$$

where $f(x) \neq 0$. The function $f(x)$ is called sometimes the *forcing term* and the equation – *forced.*

The general recipe for solving this type of equations is to consider the general solution of the corresponding *homogeneous* equation (setting $f(x) = 0$) and to find any solution of the *non-homogeneous* equation. Then, the general solution of Eq. (119) is equal to the sum of the general solution of the corresponding homogeneous equation and a particular solution of the Eq. (119).

Consider the general solution of the non-homogeneous equation y and a particular solution y_p. Let us take the difference of these two functions $z = y - y_p$ and calculate the left-hand side of the non-homogeneous equation.

$$\begin{aligned}\frac{d^2z}{dx^2} &+ a\frac{dz}{dx} + bz \\ &= \frac{d^2}{dx^2}(y - y_p) + a\frac{d}{dx}(y - y_p) + b(y - y_p) \\ &= \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by - \left(\frac{d^2y_p}{dx^2} + a\frac{dy_p}{dx} + by_p\right) \\ &= f(x) - f(x) = 0 \end{aligned} \tag{120}$$

The expression $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by$ is equal to $f(x)$ because we assumed that the function $y(x)$ is the general solution of the non-homogeneous equation. The expression $\frac{d^2y_p}{dx^2} + a\frac{dy_p}{dx} + by_p$ is also equal to $f(x)$ because we assumed that the function $y_p(x)$ is a particular solution of the non-homogeneous equation. As a result we obtained that the difference of the general solution and a particular solution satisfies the corresponding *homogeneous* equation (in Eq. (120) the right-hand side is equal to 0 as it should be in homogeneous equation). Thus we can say that the general solution of the *non-homogeneous* equation $y(x)$ is equal to the sum of a particular solution $y_p(x)$ and the general solution of the *homogeneous* equation: $y = y_p + z$. A particular solution is also called *particular integral*. The general solution of the homogeneous equation is called in this case *complementary function*.

Thus, to solve a non-homogeneous equation, we have to find the general solution of the corresponding homogeneous equation (setting $f(x) = 0$ – we know how to do this) and any particular solution of the non-homogeneous equation and to add them together. Now we have to consider the ways of finding a particular solution of the non-homogeneous equations (particular integrals). We will consider special cases for the forcing term $f(x)$.

1. $f(x)$ is a polynomial of degree n. In this case a particular integral can be found in the form of a polynomial of degree n with coefficients which can be obtained by substituting the solution in the original equation.

2. $f(x)$ is an exponential function: $f(x) = ce^{kx}$.

(i) The auxiliary equation $m^2 + am + b = 0$ has two real roots $m_{1,2}$.

a) $k \neq m_1$ and $k \neq m_2$.

Then a particular solution can be found in the form of an exponential function $y_p = Ae^{kx}$, where coefficient A can be obtained by substituting the solution in the original equation.

b) $k = m_1$ or $k = m_2$.

Then a particular solution can be found in the form of $y_p = Axe^{kx}$.

(ii) The auxiliary equation has one real root m.

a) $k \neq m$.

Then a particular solution can be found in the form of $y_p = Ae^{kx}$.

b) $k = m$.

Then a particular solution can be found in the form of $y_p = Ax^2e^{kx}$.

(iii) The auxiliary equation has no real roots.

Then a particular solution can be found in the form of $y_p = Ae^{kx}$.

In all cases the coefficient A can be found by substituting the solution in the original equation.

3. $f(x) = c\cos(\omega x) + d\sin(\omega x)$

(i) If $a \neq 0$, or $a = 0$ but $b \neq \omega^2$, then a particular solution can be found in the form $y_p = A\cos(\omega x) + B\sin(\omega x)$

(ii) If $a = 0$ and $b = \omega^2$, then a particular solution can be found in the form $y_p = x(A\cos(\omega x) + B\sin(\omega x))$

The coefficients A and B in the solutions can be found by substituting the solution in the original equation.

Another important theorem which should help in solving complicated differential equations:

If y_1 is a particular solution of equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f_1(x)$ and y_2 is a particular solution of equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f_2(x)$, then $y_p = Ay_1 + By_2$ is a particular solution of equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = Af_1(x) + Bf_2(x)$

Substituting the suggested solution $y_p = Ay_1 + By_2$ in the left-hand side of equation:

$$\begin{aligned}
\frac{d^2y_p}{dx^2} &+ a\frac{dy_p}{dx} + by_p \\
&= \frac{d^2}{dx^2}(Ay_1 + By_2) + a\frac{d}{dx}(Ay_1 + By_2) + b(Ay_1 + By_2) \\
&= A\frac{d^2y_1}{dx^2} + aA\frac{dy_1}{dx} + bAy_1 + B\frac{d^2y_2}{dx^2} + aB\frac{dy_2}{dx} + bBy_2 \\
&= A\left(\frac{d^2y_1}{dx^2} + a\frac{dy_1}{dx} + by_1\right) + B\left(\frac{d^2y_2}{dx^2} + a\frac{dy_2}{dx} + by_2\right) \\
&= Af_1(x) + Bf_2(x)
\end{aligned} \tag{121}$$

This theorem may help to solve equations where $f(x)$ is given by hyperbolic function(s) $\sinh x$ or $\cosh x$. Just recall that these functions are the sums of two exponential functions. Then two equations (with $f(x) = Ae^{kx}$) can be solved separately and the results combined according to the theorem above. The theorem can also be used when the forcing term $f(x)$ is equal to the sum of the several terms, each of them being either a polynomial or an exponential or a trigonometric function. Then we can obtain a particular solution for each of these terms and add them together.

See also Recommended textbook, Volume two, Chapter 20: Differential equations, Pages 796-809.

6 Harmonic oscillations

Here we will consider an example of *harmonic oscillations* or *harmonic motion* already mentioned in Unit 2. Suppose we have an equation

$$\frac{d^2y}{dx^2} + \omega^2 y = 0 \tag{122}$$

This is the standard way we write an equation where $y(x)$ is the function of argument x. In a particular case of the harmonic motion we can change this to more convenient variables: t – time as an argument and some function of time $x(t)$.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{123}$$

This function may be the distance from a certain initial point or the charge through electric circuit or something else. Note that the Eq. (85) is similar to Eq. (123) if $K = 0$, $F(t) = 0$ and $\omega^2 = \frac{s}{m}$. Eq. (87) is similar to Eq. (123) if $R = 0$ (or at least $R << L$), $V(t) = 0$ and $\omega^2 = \frac{1}{LC}$.

The auxiliary equation is $m^2 + \omega^2 = 0$ or $m^2 = -\omega^2$. The roots are complex: $m_{1,2} = \pm i\omega$ and the solution of the Eq. (123) is given by Eq. (111): $x = (A\cos\omega t + B\sin\omega t)$, or Eq. (112): $x = C\cos(\omega t + \phi)$, where $C = \sqrt{A^2 + B^2}$ and $\phi = \arccos(A/C)$, $\phi = \arcsin(-B/C)$. Note that the exponential factor is equal to 1 since the coefficient $a = 0$. This solution is similar to the function which was mentioned in Unit 2 as harmonic motion. It is also called linear oscillator. The graph of this function is shown in Figure 1 (dashed curve) for $\omega^2 = 8$ and initial conditions $x(0) = 1$ and $x'(0) = 0$.

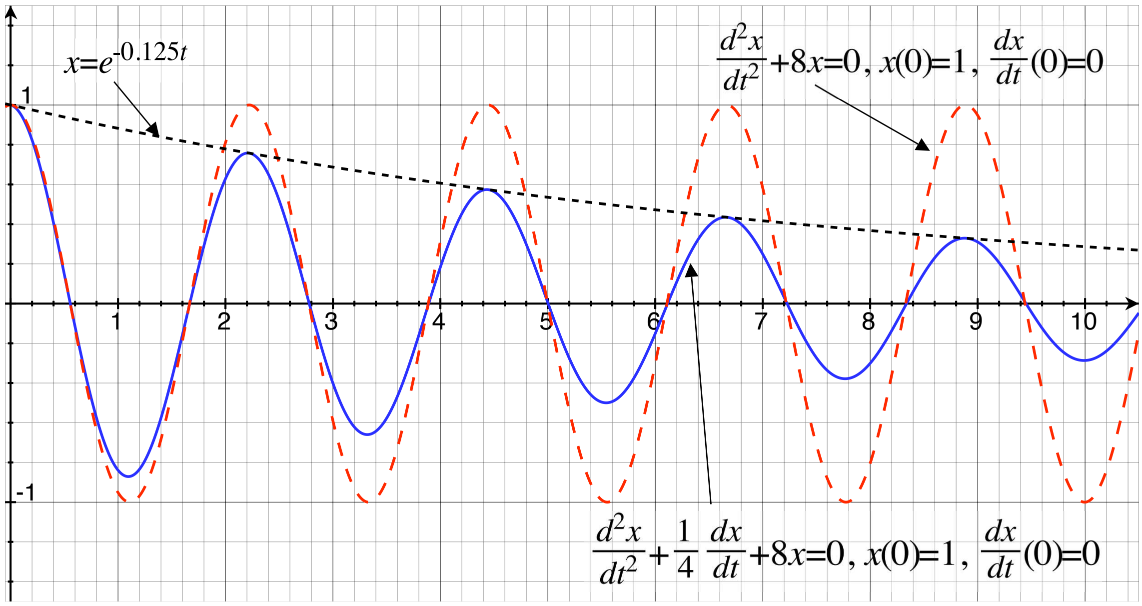

Figure 1: Examples of linear oscillator – harmonic motion

If the original equation has a term proportional to the first derivative, for example

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0 \tag{124}$$

(look for the similarities with Eqs. (85): $K/m = 2k$, and (87)) then the auxiliary equation $m^2 + 2km + \omega^2 = 0$ has the roots $m_{1,2} = -k \pm \sqrt{k^2 - \omega^2}$. Let us first assume that $k^2 < \omega^2$. Then the roots are complex $m_{1,2} = -k \pm i\sqrt{\omega^2 - k^2}$ and the solution of the Eq. (124) is given by

$$x(t) = Ce^{-kt}\cos(t\sqrt{\omega^2 - k^2} + \phi) \tag{125}$$

where the values of C and ϕ can be found if two initial conditions are given. The solution of the Eq. (124) is plotted in Figure 1 (solid curve) for $\omega^2 = 8$, $k = 1/8$ and initial conditions $x(0) = 1$ and

$x'(0) = 0$. This behaviour is called *damped free (or unforced) linear oscillator*, where the damping factor $2k$ can be the friction of the surface or the viscosity (resistance) of the medium.

If $k^2 > \omega^2$ (large friction), then we have a case of heavy damping without oscillations and the solution is exponential (see Figure 2 – solid line), as shown by Eq. (94). Figure 2 shows the case of heavy damping (solid curve): $k = 3 \to k^2 = 9 > \omega^2 = 4$ and the intermediate case with fast damping of oscillations (dashed curve): $k = 1.5 \to k^2 = 2.25 < \omega^2 = 4$.

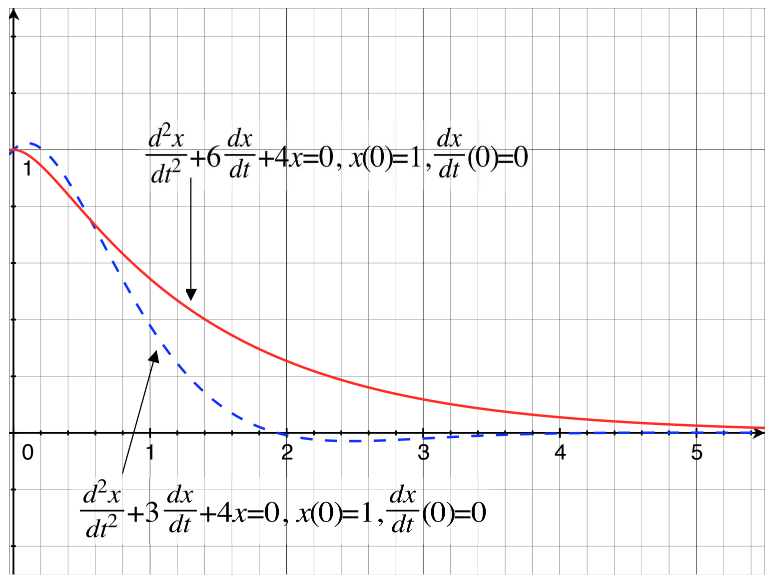

Figure 2: Examples of heavily damped oscillations

We can also consider the case of forced oscillations by adding the forcing term to the right-hand side of Eq. (124):

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = f(t) \tag{126}$$

The most interesting case is when the forcing term $f(t) = a\cos(\omega_f t)$ and $k^2 < \omega^2$:

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = a\cos(\omega_f t) \tag{127}$$

Considering the case of a mass on a spring we can say that the mass is subject to oscillating force. Without this force the mass would oscillate according to the solution given by Eq. (125) with damping and gradually come to rest. However, the external force will try to make it to oscillate with a different frequency ω_f. The result can be obtained using the method described above for non-homogeneous equations. It is equal to the sum of a particular solution and a complementary function (general solution of the corresponding homogeneous equation).

The general solution of the corresponding homogeneous equation is given by Eq. (125). A particular solution of the non-homogeneous equation is given by the formula below (you can follow the method of solving non-homogeneous equations described in the previous Section – case 3, and convert the sum of the cos and sin functions into a single cos function, or look for another method of solving such equations using complex solutions)

$$x(t) = \frac{a\cos(\omega_f t + \phi_f)}{\sqrt{(\omega^2 - \omega_f^2)^2 + (2k\omega_f)^2}} \tag{128}$$

where $\phi_f = \arccos\left(\frac{\omega^2-\omega_f^2}{\sqrt{(\omega^2-\omega_f^2)^2+(2k\omega_f)^2}}\right)$ and $\phi_f = \arcsin\left(\frac{-2k\omega_f}{\sqrt{(\omega^2-\omega_f^2)^2+(2k\omega_f)^2}}\right)$. The angle ϕ is also the polar angle of the point $((\omega^2 - \omega_f^2), -2k\omega_f)$ on an Argand diagram (see Unit 3 on complex numbers).

So the general solution of the Eq. (127) is given by

$$x(t) = \frac{a\cos(\omega_f t + \phi_f)}{\sqrt{(\omega^2 - \omega_f^2)^2 + (2k\omega_f)^2}} + Ce^{-kt}\cos(t\sqrt{\omega^2 - k^2} + \phi) \tag{129}$$

where C and ϕ are arbitrary constants that can be found if two initial conditions are given.

Figure 3 shows the function describing the general solution of the Eq. (127) with $k = \frac{1}{8}$, $\omega^2 = 8$, $a = 1$ and $\omega_f = 3$ (solid curve). The functions describing the general solution of the corresponding homogeneous equation and the particular solution (particular integral) of the non-homogeneous equations are also shown by the dotted and dashed lines, respectively.

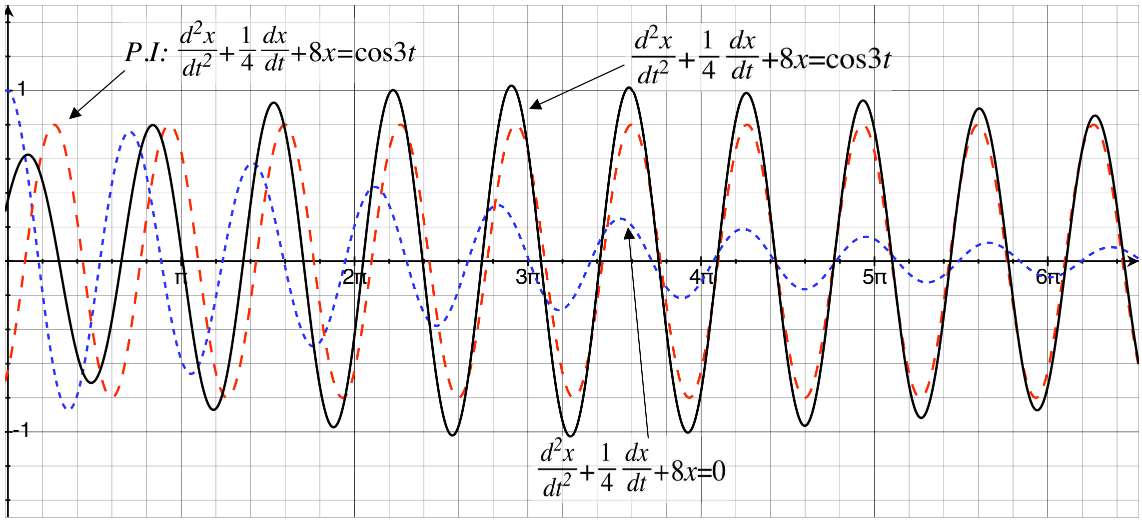

Figure 3: Harmonic functions: Forced oscillations

The general features of the forced oscillations are:

1. Free oscillations (the general solution of the homogeneous equation without forcing term, 2nd term in Eq. (129)) are not affected by the forcing term. The constants C and ϕ can be found if initial conditions are given.

2. Forced oscillations are harmonic and have the same frequency as the forcing term but different phase and amplitude. The forcing term does not depend on initial conditions.

3. Free oscillations die away due to the exponential damping factor. So all solutions finally settle into the same steady oscillations with the frequency determined by the forcing term independently of the initial conditions.

4. Oscillations with large amplitude (resonances) can be generated if $\omega_f^2 = \omega^2 - 2k^2$.