University of Sheffield
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PHY120 — Unit 3: Complex Numbers

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Contents

Learning outcomes

Understand definition of a complex number.
Be able to find real and imaginary part, modulus and phase of complex numbers.
Be able to add, subtract, multiply and divide complex numbers.
Be able to plot complex numbers on Argand diagram.
Know elementary complex functions, including polynomials and rational functions, exponentials, trigonometric and hyperbolic functions, logarithms, powers.
Know and be able to apply de Moivre's theorem and Euler's formulas.
Be able to compute logarithms, powers and roots of complex numbers.
Be able to apply complex impedance techniques to analyze simple electric circuits.

1 Complex Numbers I : Fundamental Operations

Complex numbers are widely used in physics. The solution of physical equations is often made simpler through the use of complex numbers. You will study examples of this, for example, when solving differential equations describing the motion of physical systems. Another particularly important application of complex numbers is in quantum mechanics, where they play a central role in representing the state, or wave function, of a quantum system. In this note I give a straightforward introduction to complex numbers and to simple functions of a complex variable. The first chapter introduces the basic definition of complex numbers and fundamental operations of addition and multiplication with complex numbers. The second chapter presents elementary functions of complex numbers, de Moivre's theorem and a few simple applications.

1.1 Why complex numbers?

The obvious first question is "Why introduce complex numbers?". The logical progression follows simply from the need to solve equations of increasing complexity. Thus we start with natural numbers $\mathcal{N}$ (positive integers) 1, 2, 3, ...

But $20 + y = 12 \quad \Rightarrow \quad y = -8 \quad \rightarrow$ integers $\mathcal{Z} \quad \ldots, -3, -2, -1, 0, 1, 2, \ldots$

But $4x = 6 \quad \Rightarrow \quad x = \frac{3}{2} \quad \rightarrow$ rationals $\mathcal{Q}$

But $x^2 = 2 \quad \Rightarrow \quad x = \sqrt{2} \quad \rightarrow$ irrationals $\quad \rightarrow \quad$ reals $\mathcal{R}$ (rationals and irrationals)

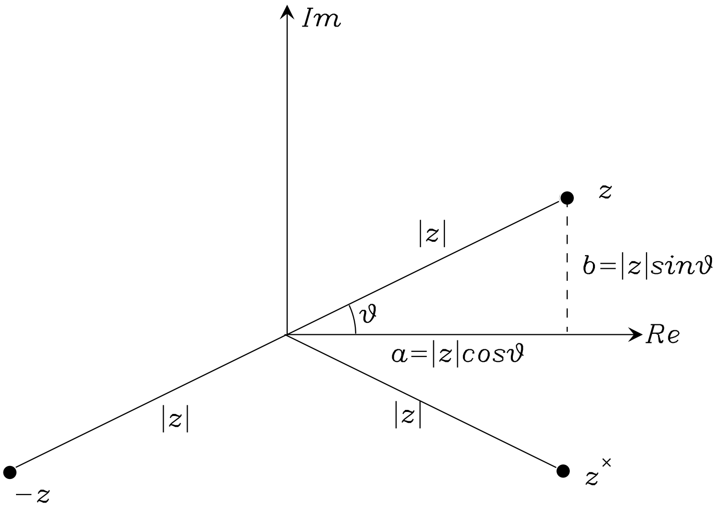
But $x^2 = -1 \quad \Rightarrow \quad x = \mathrm{i} \quad \rightarrow$ complex numbers $\mathcal{C}$

Multiples of i are called **pure imaginary** numbers. A general complex number is the sum of a multiple of 1 and a multiple of i such as $z = 2 + 3\mathrm{i}$. We often use the notation $z = a + \mathrm{i}b$, where a and b are real. (Sometimes the symbol j instead if i is used - for example in circuit theory where i is reserved for a current.)

We define operators for extracting a, b from z: $a \equiv \Re\mathrm{e}(z)$, $b \equiv \Im\mathrm{m}(z)$. We call a the **real part** and b the **imaginary part** of z.

1.2 Argand diagram

A complex number, being specified by a pair of real numbers, can be represented in the (x,y) plane. Thus the complex number $z = a + \mathrm{i}b \;\rightarrow\;$ point (a, b) in the "complex" plane (or "Argand diagram"):



Using polar co-ordinates the point (a, b) can equivalently be represented by its (r, θ) values. Thus with $\arg(z) \equiv \theta = \arctan(b/a)$ we have

$$z = |z|(\cos\theta + \mathrm{i}\sin\theta) \equiv r(\cos\theta + \mathrm{i}\sin\theta). \tag{1.1}$$

Note that the **length** or **modulus** of the vector from the origin to the point (a, b) is given by

$$|z| \equiv r = \sqrt{a^2 + b^2}. \tag{1.2}$$

θ is the argument and $|z|$ is the modulus of complex number z.

As we will show in the next section, $\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}$, the exponential of a *complex* argument. So an equivalent way of writing the polar form is

$$z = r\mathrm{e}^{\mathrm{i}\theta}. \tag{1.3}$$

It is important to get used to this form as it proves to be very useful in many applications. In Eq (1.3) we refer to r as the magnitude and to θ as the phase of complex number z. Note that there are an infinite number of values of θ which give the same values of $\cos\theta$ and $\sin\theta$ because adding an integer multiple of 2π to θ does not change them. Often one gives only one value of θ when specifying the complex number in polar form but, as we shall see, it is important to include this ambiguity when for instance taking roots or logarithms of a complex number.

It also proves useful to define the **complex conjugate** z^* (or $\bar{z}$) of z by reversing the sign of the imaginary part, i.e.

$$z^* \equiv a - \mathrm{i}b. \tag{1.4}$$

The complex numbers z^* and $-z$ are also shown in the figure. We see that taking the complex conjugate z^* of z can be represented by reflection with respect to the real axis.

Example 1.1

Express $z \equiv a + ib = -1 - i$ in polar form.

Here $r = \sqrt{2}$ and $\arctan(b/a) = \arctan 1 = \pi/4$. However it is necessary to identify the correct quadrant for θ. Since a and b are both negative so too are $\cos\theta$ and $\sin\theta$. Thus θ lies in the third quadrant $\theta = \frac{5\pi}{4} + 2n\pi$ where n is any positive or negative integer. Thus finally we have $z = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{5\pi}{4}+2n\pi}, n = 0, \pm1, \pm2, \cdots$, where we have made the ambiguity in the phase explicit.

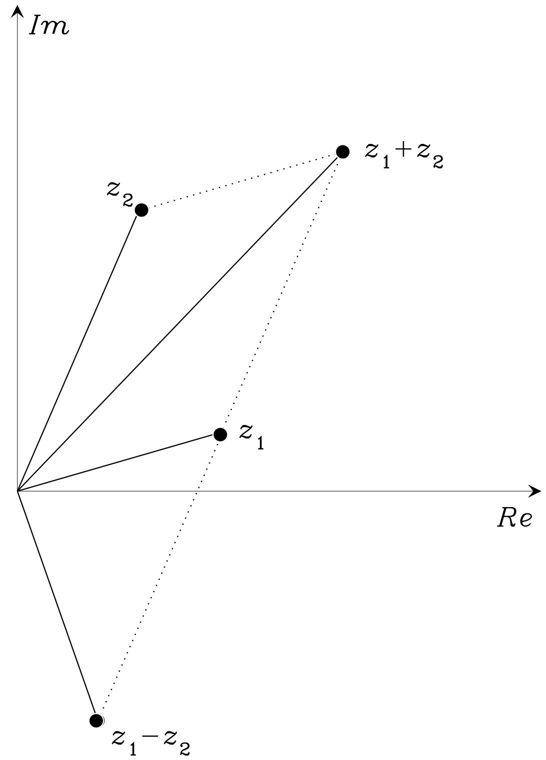
In the next two subsections we describe the basic operations of addition and multiplication with complex numbers.

1.3 Addition and subtraction

Addition and subtraction of complex numbers follow the same rules as for ordinary numbers except that the real and imaginary parts are treated separately:

$$z_1 \pm z_2 \equiv (a_1 \pm a_2) + \mathrm{i}(b_1 \pm b_2) \tag{1.5}$$

Since the complex numbers can be represented in the Argand diagram by vectors, addition and subtraction of complex numbers is the same as addition and subtraction of vectors as is shown in the figure. Adding z_2 to any z amounts to translating z by z_2.



Example 1.2

Find $z_1 + z_2$ and $z_1 - z_2$ for $z_1 = 2e^{i\pi/4}$, $z_2 = e^{-3i\pi/4}$.

We have $z_1 \pm z_2 = 2(1+i)/\sqrt{2} \mp (1+i)/\sqrt{2}$, so $z_1 + z_2 = (1+i)/\sqrt{2}$, and $z_1 - z_2 = 3(1+i)/\sqrt{2}$.

1.4 Multiplication and division

Remembering that $\mathrm{i}^2 = -1$ it is easy to define multiplication for complex numbers :

$$\begin{aligned} z_1 z_2 &= (a_1 + \mathrm{i}b_1)(a_2 + \mathrm{i}b_2) \\ &\equiv (a_1 a_2 - b_1 b_2) + \mathrm{i}(a_1 b_2 + b_1 a_2) \end{aligned} \tag{1.6}$$

Example 1.3

Find the product $z_1 z_2$ of complex numbers $z_1 = 1 + i$, $z_2 = -3 + 2i$.

Here $z_1 z_2 = -3 - 2 + 2i - 3i = -5 - i$.

Note that the product of a complex number and its complex conjugate, $|z|^2 \equiv zz^* = (a^2 + b^2)$, is real (and ≥ 0) and, c.f. eq (1.2), is given by the square of the length of the vector representing the complex number $zz^* \equiv |z|^2 = (a^2 + b^2)$.

It is necessary to define division also. This is done by multiplying the numerator and denominator of the fraction by the complex conjugate of the denominator :

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} \tag{1.7}$$

One may see that division by a complex number has been changed into multiplication by a complex number. The denominator in the right hand side of eq (1.7) has become a real number and all we now need to define complex division is the rule for multiplication of complex numbers.

Multiplication and division are particularly simple when the polar form of the complex number is used. If $z_1 = |z_1|\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = |z_2|\mathrm{e}^{\mathrm{i}\theta_2}$, then their product is given by

$$z_1 * z_2 = |z_1| * |z_2|\mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)}. \tag{1.8}$$

To multiply any z by $z_2 = |z_2|\mathrm{e}^{\mathrm{i}\theta_2}$ means to rotate z by angle θ_2 and to dilate its length by $|z_2|$.

To determine $\frac{z_1}{z_2}$ note that

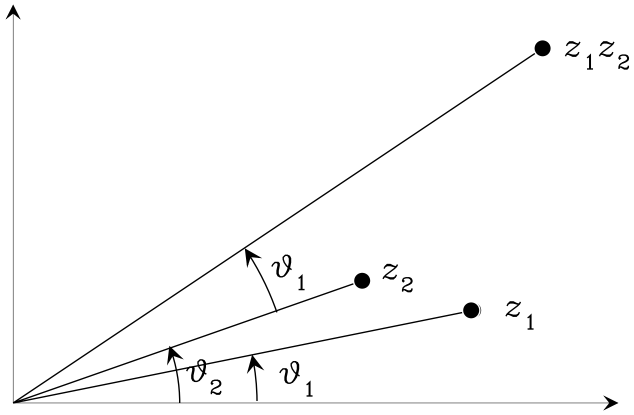
$$\begin{aligned} z &= |z|(\cos\theta + \mathrm{i}\sin\theta) = |z|\mathrm{e}^{\mathrm{i}\theta} \\ z^* &= |z|(\cos\theta - \mathrm{i}\sin\theta) = |z|\mathrm{e}^{-\mathrm{i}\theta} \\ \frac{1}{z} &= \frac{z^*}{zz^*} = \frac{\mathrm{e}^{-\mathrm{i}\theta}}{|z|}. \end{aligned} \tag{1.9}$$

Thus

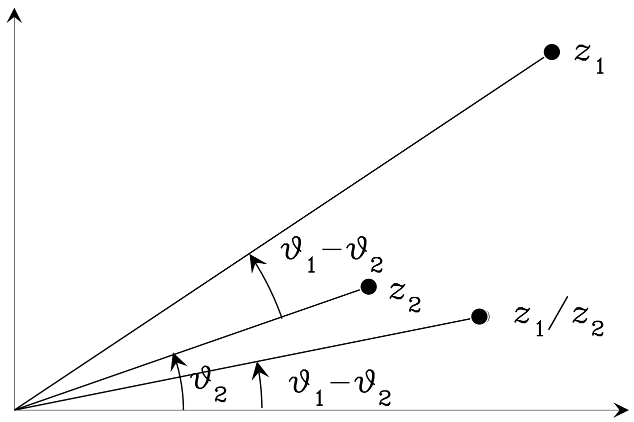
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{|z_1|\mathrm{e}^{\mathrm{i}\theta_1} * \mathrm{e}^{-\mathrm{i}\theta_2}}{|z_2|} \\ &= \frac{|z_1|}{|z_2|}\mathrm{e}^{\mathrm{i}(\theta_1 - \theta_2)} \end{aligned} \tag{1.10}$$

1.4.1 Graphical representation of multiplication & division

$$z_1 z_2 = |z_1||z_2|\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$$



$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}$$



Example 1.4

Find the modulus $|z_1/z_2|$ when $\begin{cases} z_1 = 1+2\mathrm{i} \\ z_2 = 1-3\mathrm{i} \end{cases}$

Clumsy method:

$$\begin{aligned} \left|\frac{z_1}{z_2}\right| &= \left|\frac{1+2\mathrm{i}}{1-3\mathrm{i}}\right| = \frac{|z_1 z_2^*|}{|z_2|^2} \\ &= \frac{|(1+2\mathrm{i})(1+3\mathrm{i})|}{1+9} = \frac{|(1-6)+\mathrm{i}(2+3)|}{10} \\ &= \frac{\sqrt{25+25}}{10} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \end{aligned}$$

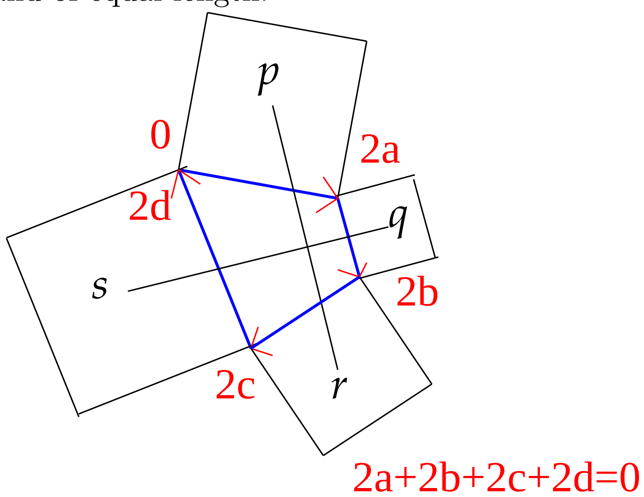
Elegant method:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{1+4}}{\sqrt{1+9}} = \frac{1}{\sqrt{2}}$$

Methods based on complex addition and multiplication can be useful to analyze plane geometry problems as in the following example.

Example 1.5

Consider an arbitrary quadrilateral and construct squares on each side as in the figure below. Show that segments joining the centres of opposite squares are perpendicular and of equal length.



Consider the complex plane and let the vertices of the quadrilateral be at points $2a$, $2a+2b$, $2a+2b+2c$, and $2a+2b+2c+2d=0$. The centre of the square on the first side is at

$$p = a + ae^{i\pi/2} = a(1+i) \ .$$

Likewise, the centres of the other squares are at

$$q = 2a + b(1+i) \ , \quad r = 2a+2b+c(1+i) \ , \quad s = 2a+2b+2c+d(1+i) \ .$$

Thus

$$A \equiv s - q = b(1-i) + 2c + d(1+i) \ , \quad B \equiv r - p = a(1-i) + 2b + c(1+i) \ .$$

A and B being perpendicular and of equal length means $B = Ae^{i\pi/2}$, i.e., $B = iA$, i.e., $A + iB = 0$. We now show that this is indeed the case:

$$A + iB = b(1-i) + 2c + d(1+i) + ia(1-i) + 2ib + ic(1+i)$$

$$= b(1+i) + c(1+i) + d(1+i) + a(1+i) = (1+i)(a+b+c+d) = 0 \ .$$

2 Complex Numbers II: Elementary Functions

This chapter is devoted to elementary functions of complex numbers, including exponential, trigonometric, hyperbolic, logarithmic functions. We give de Moivre's theorem and show examples of its uses. We discuss powers and roots of complex numbers.

2.1 Elementary functions of complex variable

We may define polynomials and rational functions of complex variable z based on the algebraic operations of multiplication and addition of complex numbers introduced in the previous section. For example, separating real and imaginary parts, $z = x + iy$, we have

$$f(z) = z^2 = (x + iy)(x + iy) = x^2 - y^2 + 2ixy \ .$$

Similarly,

$$f(z) = \frac{1}{z} = \frac{z^*}{|z|^2} = \frac{x}{x^2 + y^2} - i \ \frac{y}{x^2 + y^2} \ .$$

To define the complex exponential and related functions such as trigonometric and hyperbolic functions, we use power series expansion.

2.1.1 The complex exponential function

The definition of the exponential, cosine and sine functions of a *real* variable can be done by writing their series expansions :

$$\begin{aligned} \mathrm{e}^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \end{aligned} \tag{2.1}$$

For small x a few of terms may be sufficient to provide a good approximation. Thus for very small x, $\sin x \approx x$.

In a similar manner we may define functions of the complex variable z. The complex exponential is defined by

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \cdots \tag{2.2}$$

A special case is if z is purely imaginary $z = \mathrm{i}\theta$. Using the fact that $\mathrm{i}^{2n} = 1$ or -1 for n even or odd and $\mathrm{i}^{2n+1} = i$ or $-i$ for n even or odd we may write

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\theta} &= \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\Big) + \mathrm{i}\Big(\theta - \frac{\theta^3}{3!} + \cdots\Big) \\ &= \cos\theta + \mathrm{i}\sin\theta \end{aligned} \tag{2.3}$$

This is the relation that we used in writing a complex number in polar form, c.f. Eq (1.3). Thus

$$\begin{aligned} z &= |z|(\cos\theta + \mathrm{i}\sin\theta) = |z|\mathrm{e}^{\mathrm{i}\theta} \\ z^* &= |z|(\cos\theta - \mathrm{i}\sin\theta) = |z|\mathrm{e}^{-\mathrm{i}\theta} \\ \frac{1}{z} &= \frac{z^*}{zz^*} = \frac{\mathrm{e}^{-\mathrm{i}\theta}}{|z|}. \end{aligned} \tag{2.4}$$

We may find a useful relation between sines and cosines and complex exponentials. Adding and then subtracting the first two of equations (2.4) we find that

$$\begin{aligned} \cos\theta &= \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) \\ \sin\theta &= \tfrac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}) \end{aligned} \tag{2.5}$$

Eqs (2.4) and (2.5) are known as **Euler's** formulas.

2.1.2 The complex sine and cosine functions

In a similar manner we can define $\cos z$ and $\sin z$ by replacing the argument x in (2.1) by the complex variable z. The analogue of Eq (2.3) is

$$\begin{aligned} \mathrm{e}^{\mathrm{i}z} &= \Big(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\Big) + \mathrm{i}\Big(z - \frac{z^3}{3!} + \cdots\Big) \\ &= \cos z + \mathrm{i}\sin z \end{aligned} \tag{2.6}$$

Similarly one has

$$\begin{aligned} \cos z &= \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) \\ \sin z &= \tfrac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) \end{aligned} \tag{2.7}$$

From this we learn that the cosine and the sine of an imaginary angle are

$$\begin{aligned} \cos(\mathrm{i}b) &= \tfrac{1}{2}(\mathrm{e}^{-b} + \mathrm{e}^{b}) = \cosh b \\ \sin(\mathrm{i}b) &= \tfrac{1}{2\mathrm{i}}(\mathrm{e}^{-b} - \mathrm{e}^{b}) = \mathrm{i}\sinh b, \end{aligned} \tag{2.8}$$

where we have used the definitions of the hyperbolic functions

$$\begin{aligned} \cosh b &\equiv \tfrac{1}{2}(\mathrm{e}^{b} + \mathrm{e}^{-b}) \\ \sinh b &\equiv \tfrac{1}{2}(\mathrm{e}^{b} - \mathrm{e}^{-b}). \end{aligned} \tag{2.9}$$

Note:

Hyperbolic functions get their name from the identity $\cosh^2\theta - \sinh^2\theta = 1$, which is readily proved from (2.9) and is reminiscent of the equation of a hyperbola, $x^2 - y^2 = 1$.

2.1.3 Complex hyperbolic sine and cosine functions

We define complex hyperbolic functions in a similar manner as done above for complex trigonometric functions, by replacing the real argument in the power series expansion by complex variable z. Then we have

$$\begin{aligned} \cosh z &= \tfrac{1}{2}(\mathrm{e}^{z} + \mathrm{e}^{-z}) \\ \sinh z &= \tfrac{1}{2}(\mathrm{e}^{z} - \mathrm{e}^{-z}). \end{aligned} \tag{2.10}$$

Example 2.1

Find the real and imaginary parts of $\cos i$ and $\sin i$.

From Eqs (2.7) we have

$$\cos i = (e^{-1} + e)/2 \ , \qquad \sin i = (e^{-1} - e)/(2i) \ .$$

So, real and imaginary parts of $\cos i$ are $(e + 1/e)/2$ and 0; real and imaginary parts of $\sin i$ are 0 and $(e - 1/e)/2$. (Equivalently, apply Eqs (2.8) with $b = 1$.)

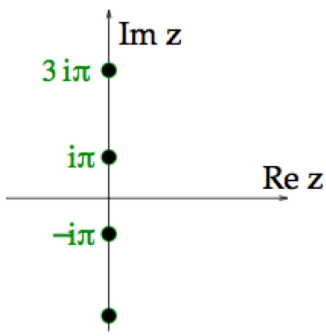
Example 2.2

Find the points z in the complex plane where $e^z = -1$.

Writing $z = x + iy$, we have

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y) = -1 = e^{i\pi(1+2n)}$$

that is, $x = 0$, $y = \pi(1 + 2n)$. Thus $z = i\pi(2n + 1)$, with n integer.



2.1.4 Curves in the complex plane

The set, or “locus”, of points satisfying some constraint on a complex parameter traces out a curve in the complex plane. For example the constraint $|z| = 1$ requires that the length of the vector from the origin to the point z is constant and equal to 1. This corresponds to the set of points lying on a circle of unit radius.

Instead of determining the geometric structure of the constraint one may instead solve the constraint equation algebraically and look for the equation of the curve. This has the advantage that the method is in principle straightforward, though the details may be algebraically involved. In Cartesian coordinates, with $z = x + iy$, the algebraic constraint corresponding to $|z| = 1$ is $|z|^2 = x^2 + y^2 = 1$ which is the equation of a circle as expected. In polar coordinates (r, θ) the calculation is even simpler, $|z| = r = 1$.

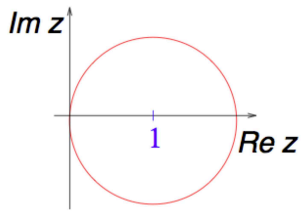
As a second example consider the constraint $|z - z_0| = 1$. This is the equation of a unit circle centre z_0 as may be immediately seen by changing the coordinate system to $z' = (z - z_0)$.

Alternatively one may solve the constraint algebraically to find $|z - z_0|^2 = (x - x_0)^2 + (y - y_0)^2 = 1$ which is the equation of the unit circle centred at the point (x_0, y_0). The solution in polar coordinates is not so straightforward in this case, showing that it is important to try the alternate forms when looking for the algebraic solution.

Example 2.3

Draw the curve in the complex z plane which is defined by $|z - 1| = 1$.

This is the set of points z in the Argand diagram whose distance from point 1 on the real axis is constant and equal to 1:



To illustrate the techniques for finding curves in the complex plane I present some further examples:

Example 2.4

What is the locus in the Argand diagram that is defined by $\left|\dfrac{z - \mathrm{i}}{z + \mathrm{i}}\right| = 1$?

Equivalently we have $|z - \mathrm{i}| = |z + \mathrm{i}|$, so the distance to z from $(0, 1)$ is the same as the distance from $(0, -1)$. Hence the solution is the "real axis".

Alternatively we may solve the equation

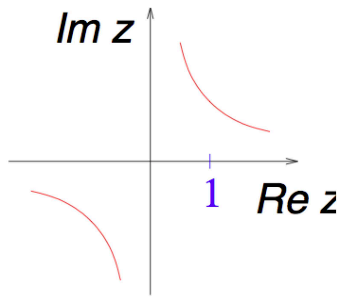
$$x^2 + (y - 1)^2 = x^2 + (y + 1)^2$$

which gives $y = 0$, x arbitrary, corresponding to the real axis.

Example 2.5

What is the locus in the Argand diagram that is defined by $\Im\mathrm{m}(z^2) = 2$?

Using $z^2 = x^2 - y^2 + 2ixy$ we have $2xy = 2$ i.e. $y = 1/x$.



2.2 de Moivre's theorem

Starting with the polar form of complex numbers and taking power n yields the general form of **de Moivre's** theorem:

$$z^n = (r\mathrm{e}^{\mathrm{i}\theta})^n = r^n \mathrm{e}^{in\theta} = r^n(\cos n\theta + i\sin n\theta) \tag{2.11}$$

for any integer n. That is,

$$(\mathrm{e}^{\mathrm{i}\theta})^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta. \tag{2.12}$$

In many problems it is easier and more compact to work with de Moivre's theorem and the complex exponential rather than with sines and cosines - see examples below.

Example 2.6

Find $(1+\mathrm{i})^8$. Taking powers is much simpler in polar form so we write $(1+\mathrm{i}) = \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}$. Hence $(1+\mathrm{i})^8 = (\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4})^8 = 16\mathrm{e}^{2\pi\mathrm{i}} = 16$.

Example 2.7

Find real and imaginary parts of i^{-5}. We have $i^{-5} = e^{-5i\pi/2} = -i$, so real and imaginary parts are 0 and -1.

2.2.1 Trigonometric identities

Eq (2.12) generates simple identities for $\cos n\theta$ and $\sin n\theta$. For example, for $n = 2$ we obtain, by equating the real and imaginary parts on the two sides of Eq (2.12),

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\cos\theta\sin\theta \end{aligned} \tag{2.13}$$

that is, the trigonometric formulas for the cosine and sine of the double angle.

The complex exponential is very useful in establishing trigonometric identities. For example, to obtain the cosine and sine of the sum of two angles a and b, consider the complex exponential $e^{i(a+b)}$. We have

$$\begin{aligned}\cos(a+b)+\mathrm{i}\sin(a+b) &= \mathrm{e}^{\mathrm{i}(a+b)} = \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b} \\ &= (\cos a+\mathrm{i}\sin a)(\cos b+\mathrm{i}\sin b) \\ &= (\cos a\cos b-\sin a\sin b)+\mathrm{i}(\cos a\sin b+\sin a\cos b)\end{aligned} \tag{2.14}$$

where we have used the property of exponentials that $\mathrm{e}^{\mathrm{i}(a+b)} = \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b}$. Here we have a complex equation relating a complex number on the left hand side (LHS) to a complex number on the right hand side (RHS). To solve it we must equate the real parts of the LHS and the RHS and separately the imaginary parts of the LHS and RHS. Note that a complex equation is equivalent to two real equations. Comparing real and imaginary parts on the two sides of (2.14), we deduce that

$$\begin{aligned}\cos(a+b) &= \cos a\cos b-\sin a\sin b \\ \sin(a+b) &= \sin a\cos b+\cos a\sin b\end{aligned}$$

Example 2.8

Prove the trigonometric identity $\sin^3\theta = [3\sin\theta-\sin 3\theta]/4$.

$$\begin{aligned}\sin^3\theta &= \left[\left(e^{i\theta}-e^{-i\theta}\right)/(2i)\right]^3 \\ &= -(1/(8i))\left[e^{3i\theta}-3e^{2i\theta}e^{-i\theta}+3e^{i\theta}e^{-2i\theta}-e^{-3i\theta}\right] \\ &= -\left[\left(e^{3i\theta}-e^{-3i\theta}\right)-3\left(e^{i\theta}-e^{-i\theta}\right)\right]/(8i) \\ &= -[\sin 3\theta-3\sin\theta]/4\end{aligned}$$

2.2.2 Uses of de Moivre's theorem in differential equations and series

De Moivre's theorem can be usefully applied to the solution of differential equations and summation of series. Here I just give two examples. You will see many more in further courses.

Example 2.9

Solving differential equations is often simpler using complex exponentials. As an introductory example I consider here the solution of simple harmonic motion, $\mathrm{d}^2y/\mathrm{d}\theta^2+y=0$. The general solution is known to be $y = A\cos\theta+B\sin\theta$ where A and B are real constants. To solve it using the complex exponential we first write $y = \Re\mathrm{e}z$ so that the equation becomes $\mathrm{d}^2\Re\mathrm{e}z/\mathrm{d}\theta^2+\Re\mathrm{e}z = \Re\mathrm{e}(\mathrm{d}^2z/\mathrm{d}\theta^2+z)=0$. The solution to the equation $d^2z/\mathrm{d}\theta^2+z=0$ is simply $z = C\mathrm{e}^{i\theta}$ where C is a (complex) constant. You may check that this is the case simply by substituting the answer in the original equation, and evaluating the derivative: $d^2(C\mathrm{e}^{i\theta})/d\theta^2 = i^2 C\mathrm{e}^{i\theta} = -C\mathrm{e}^{i\theta}$. Writing $C = A - \mathrm{i}B$ one finds

$$\begin{aligned} y &= \Re\mathrm{e} z = \Re\mathrm{e}((A - \mathrm{i}B)(\cos\theta + \mathrm{i}\sin\theta)) \\ &= A\cos\theta + B\sin\theta \end{aligned} \tag{2.15}$$

Thus we have derived the general solution in one step - there is no need to look for the sine and cosine solutions separately. Although the saving in effort through using complex exponentials is modest in this simple example, it becomes significant in the solution of more general differential equations.

Example 2.10

Series involving sines and cosines may often be summed using de Moivre. As an example we will prove that for $0 < r < 1$

$$\sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \frac{(1+r)\sin\theta}{1 - 2r\cos 2\theta + r^2}$$

Proof:

$$\begin{aligned} \sum_{n=0}^{\infty} r^n \sin(2n+1)\theta &= \sum_n r^n \Im\mathrm{m}(\mathrm{e}^{\mathrm{i}(2n+1)\theta}) = \Im\mathrm{m}\Big(\mathrm{e}^{\mathrm{i}\theta}\sum_n (r\mathrm{e}^{2\mathrm{i}\theta})^n\Big) \\ &= \Im\mathrm{m}\left(\mathrm{e}^{\mathrm{i}\theta}\frac{1}{1 - r\mathrm{e}^{2\mathrm{i}\theta}}\right) \\ &= \Im\mathrm{m}\left(\frac{\mathrm{e}^{\mathrm{i}\theta}(1 - r\mathrm{e}^{-2\mathrm{i}\theta})}{(1 - r\mathrm{e}^{2\mathrm{i}\theta})(1 - r\mathrm{e}^{-2\mathrm{i}\theta})}\right) \\ &= \frac{\sin\theta + r\sin\theta}{1 - 2r\cos 2\theta + r^2} \end{aligned}$$

2.2.3 Identities for complex sines and cosines

We may use the result of (2.7) to evaluate the cosine of a complex number:

$$\begin{aligned} \cos z &= \cos(a + \mathrm{i}b) \\ &= \tfrac{1}{2}(\mathrm{e}^{(\mathrm{i}a - b)} + \mathrm{e}^{(-\mathrm{i}a + b)}) \\ &= \tfrac{1}{2}(\mathrm{e}^{-b}(\cos a + \mathrm{i} sina) + \mathrm{e}^{b}(\cos a - \mathrm{i}\sin a)) \\ &= \cos a\cosh b - \mathrm{i}\sin a\sinh b. \end{aligned} \tag{2.16}$$

Analogously

$$\sin z = \sin a\cosh b + \mathrm{i}\cos a\sinh b. \tag{2.17}$$

2.3 Complex logarithm and complex powers

In this section we see examples of “multi-valued” functions, that is, functions which associate to each value of z not just one but a whole set of well-prescribed values. The logarithm is one such function. In the next section we will study other such examples, the roots of complex numbers.

2.3.1 The logarithm

The logarithmic function $f(z) = \ln z$ is the inverse of the exponential function meaning that if one acts on z by the logarithmic function and then by the exponential function one gets just z, $\mathrm{e}^{\ln z} = z$. We may use this property to define the logarithm of a complex variable :

$$\mathrm{e}^{\ln z} = z = |z|e^{\mathrm{i}\theta} = \mathrm{e}^{\ln |z|}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\ln |z| + \mathrm{i}\theta}$$
$$\Rightarrow \quad \underset{(a)}{\ln z = \ln |z|} + \underset{(b)}{\mathrm{i}\arg(z)} \tag{2.18}$$

Part (a) is just the normal logarithm of a real variable and gives the real part of the logarithmic function while part (b) gives its imaginary part. Note that the infinite ambiguity in the phase of z is no longer present in $\ln z$ because the addition of an integer multiple of 2π to the argument of z changes the imaginary part of the logarithm by the same amount. Thus it is essential, when defining the logarithm, to know precisely the argument of z. We can rewrite Eq (2.18) explicitly as

$$\ln z = \ln |z| + \mathrm{i}(\theta + 2\pi n) \ , \quad n \ \text{integer} \quad . \tag{2.19}$$

For different n we get different values of the complex logarithm. So we need to assign n to fully specify the logarithm.

The different values corresponding to different n are called "branches" of the logarithm. $n = 0$ is called the principal branch.

A function of z which may take not one but multiple values for a given value of z is called multi-valued. The logarithm is our first example of a multi-valued function.

Example 2.11

Find all values of $\ln(-1)$.

$$\ln(-1) = \ln e^{i\pi} = \ln 1 + i(\pi + 2\pi n) = i\pi + 2\pi i n \ , \quad n \ \text{integer} \quad .$$

For the principal branch $n = 0$

$$\ln(-1) = i\pi \qquad (n = 0) \quad .$$

Note:

$e^{\ln z}$ always equals z, while $\ln e^z$ does not always equal z.

Let $z = a + ib = re^{i\theta}$. Then $\ln z = \ln r + i(\theta + 2\pi n)$, n integer. So

$$e^{\ln z} = e^{\ln r + i(\theta + 2\pi n)} = re^{i\theta} \underbrace{e^{2\pi n i}}_{1} = re^{i\theta} = z \quad .$$

On the other hand $e^z = e^{a+ib} = e^a e^{ib}$. Therefore

$$\ln e^z = \ln e^a + i(b + 2\pi n) = \underbrace{a + ib}_{z} + 2\pi i n = z + 2\pi i n \ \text{ which may be } \neq z \quad .$$

Example 2.12
Find the real and imaginary parts of the principal branch of the logarithm of $z = 1 + i$.

$$(1+i) = \sqrt{2}e^{i\pi/4} \ \Rightarrow \ \ln(1+i) = 2^{-1}\ln 2 + i\pi/4 \quad (n=0)$$

2.3.2 Complex powers

Once we have the complex logarithm, we can define complex powers $f(z) = z^{\alpha}$, where both z and α are complex:

$$f(z) = z^{\alpha} = e^{\alpha \ln z} \quad . \tag{2.20}$$

Since the logarithm is multi-valued, complex powers also are multi-valued functions.

Example 2.13
Show that i^i is real and the principal-branch value is $i^i = 1/\sqrt{e^{\pi}}$.

$$i^i = e^{i \ln i} = e^{i[\ln 1 + i(\pi/2 + 2\pi n)]} = e^{-\pi/2 - 2\pi n}.$$

These values are all real. For $n = 0$ we have $i^i = e^{-\pi/2} = 1/\sqrt{e^{\pi}}$.

2.4 Roots of complex numbers

A number u is defined to be an n-th root of complex number z if $u^n = z$. Then we write $u = z^{1/n}$. The following result obtains.

Every complex number has exactly n distinct n-th roots.

Proof. Let $z = r(\cos\theta + i\sin\theta)$; $u = \rho(\cos\alpha + i\sin\alpha)$. Then, using de Moivre's theorem,

$$r(\cos\theta + i\sin\theta) = \rho^n(\cos\alpha + i\sin\alpha)^n = \rho^n(\cos n\alpha + i\sin n\alpha)$$

$$\Rightarrow \ \rho^n = r \ , \quad n\alpha = \theta + 2\pi k \qquad (k \text{ integer})$$

$$\text{Therefore } \rho = r^{1/n} \ , \quad \alpha = \theta/n + 2\pi k/n \ .$$

We thus see that we get n distinct values for k, from 0 to $n-1$, corresponding to n distinct values of u for $z \neq 0$. So

$$u = z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right], \ k = 0, 1, \ldots, n-1 \ . \tag{2.21}$$

We note that the function $f(z) = z^{1/n}$ is a multi-valued function.

Example 2.14

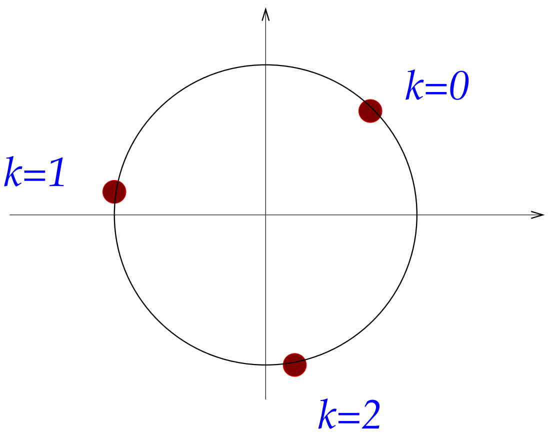
Find all cubic roots of $z = -1 + i$.

Applying Eq (2.21) we have

$$u = (-1+i)^{1/3}$$
$$= (\sqrt{2})^{1/3}\left[\cos\left(\frac{3\pi}{4}\frac{1}{3} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{3\pi}{4}\frac{1}{3} + \frac{2\pi k}{3}\right)\right] \quad , \quad k = 0, 1, 2 \ .$$

That is, the three cubic roots of $-1 + i$ are

$$2^{1/6}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad (k = 0) \quad ,$$
$$2^{1/6}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \quad (k = 1) \quad ,$$
$$2^{1/6}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) \quad (k = 2) \quad .$$



Equivalently, using Eq (2.20),

$$u = (-1+i)^{1/3} = e^{(1/3)\ln(-1+i)} = e^{(1/3)[\ln\sqrt{2} + i(3\pi/4 + 2k\pi)]}$$
$$= (\sqrt{2})^{1/3} e^{i(\pi/4 + 2k\pi/3)} \quad .$$

Example 2.15

Consider the equation

$z^n = 1 \quad \Rightarrow \quad z = 1^{1/n}$

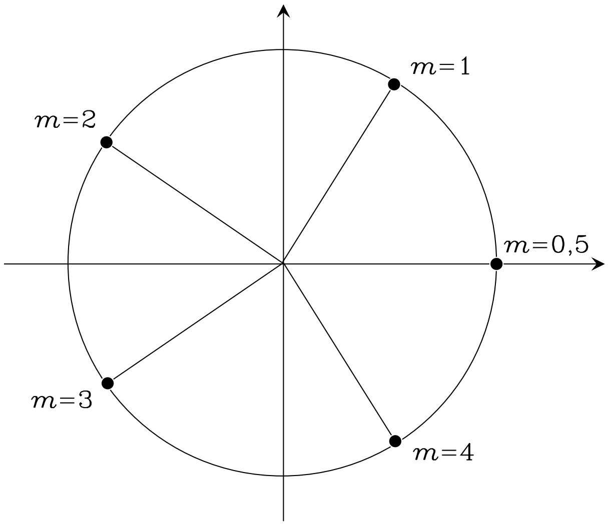
The solution is given by the nth roots of unity. In taking roots it is crucial to allow for the ambiguity in the phase of a (complex) number

$$1 = \mathrm{e}^{2m\pi\mathrm{i}} \quad \Rightarrow \quad 1^{1/n} = \mathrm{e}^{2m\pi\mathrm{i}/n}$$
$$= \cos\left(\frac{2m\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2m\pi}{n}\right) \qquad (2.22)$$

E.g. for $n = 5$

$$1^{1/5} = \cos\left(\frac{2m\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2m\pi}{5}\right) \quad (m = 0, 1, 2, 3, 4).$$

The roots may be drawn in the Argand plane and correspond to five equally spaced points in the plane :



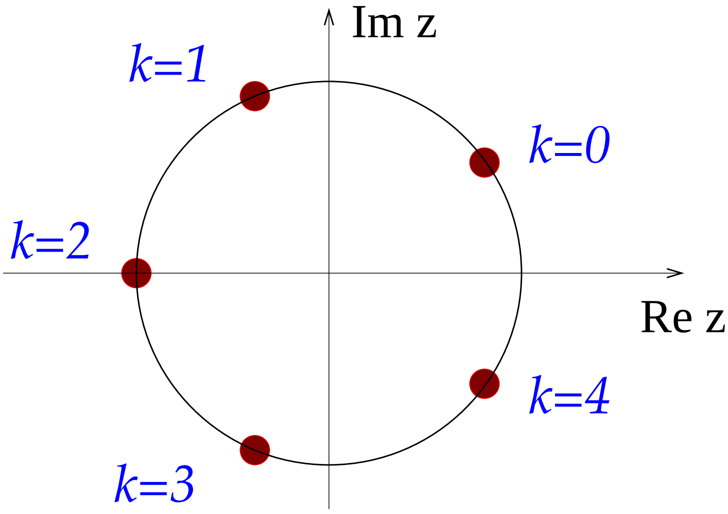
Example 2.16
Consider the equation

$$z^5 + 32 = 0\ .$$

This is similar to the previous case. The solutions are the fifth roots of -32:

$$(-32)^{1/5} = 32^{1/5}\left[\cos\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2\pi k}{5}\right)\right] \quad , \quad k = 0, 1, 2, 3, 4$$

that is,
$k = 0:\ \ 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$
$k = 1:\ \ 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$
$k = 2:\ \ -2$
$k = 3:\ \ 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$
$k = 4:\ \ 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$



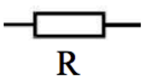
2.5 Application: complex impedences in electric circuit theory

If an alternating current $I = I_0 \cos \omega t$, where I_0 is the current amplitude and ω is the frequency, flows through an element of an electric circuit such as a resistor, an inductor or a capacitor, the voltage V across that element is not necessarily in phase with the current. More precisely, as we will see shortly, it is in-phase for a resistor, but out-of-phase for an inductor or a capacitor. The voltage amplitude, on the other hand, is proportional to the current amplitude I_0 via a proportionality coefficient which may or may not be independent of the frequency ω. We will see in a moment that it is ω-independent for a resistor, but ω-dependent for an inductor or a capacitor.

Then, since these linear voltage-current relationships in ac circuits involve non-trivially both a *magnitude* and a *phase*, it turns out to be extremely convenient to describe these relationships via complex numbers - see e.g. notation introduced at the outset in Eq (1.3) - and use complex exponential representations for the electric signals, rather than sines and cosines. The quantity which expresses the relationship between voltage V and current I is termed **complex impedence** Z and is defined by

$$V = IZ. \tag{2.23}$$

Let us consider one by one the cases of the circuit elements mentioned above. Suppose first the current $I = I_0 \cos \omega t$ flows through a resistor of resistance R.

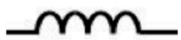


The voltage across the resistor is set by Ohm's law to be

$$V = IR = I_0 R \cos \omega t. \tag{2.24}$$

The voltage amplitude is $V_0 = I_0 R$, and its phase coincides with that of the current.

Next suppose the resistor is replaced by an inductor of inductance L. In this case



too there is a voltage across the inductor which is now given, using the laws of magnetic induction, by

$$\begin{aligned} V &= L dI/dt = -I_0 \omega L \sin \omega t \\ &= I_0 \omega L \cos(\omega t + \pi/2). \end{aligned} \tag{2.25}$$

The voltage amplitude V_0 is thus now related to the current amplitude I_0 by a coefficient which depends both on the inductance L and on the frequency ω, $V_0 = I_0 \omega L$. The voltage now differs in phase from the current: it *leads* the current by $\pi/2$.

Finally consider an alternating source on a capacitor of capacitance C. The

voltage across the capacitor may be computed by integrating the current as

$$\begin{aligned} V = \frac{1}{C} \int I dt &= I_0 \frac{1}{\omega C} \sin \omega t \\ &= I_0 \frac{1}{\omega C} \cos(\omega t - \pi/2). \end{aligned} \tag{2.26}$$

The voltage amplitude is thus $V_0 = I_0/(\omega C)$. The voltage phase differs from that of the current: the voltage now *lags* the current by $\pi/2$.

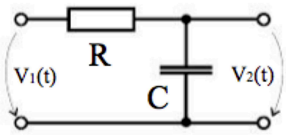
The results in Eqs (2.24), (2.25) and (2.26) can be put in the compact form of (2.23) if we use complex formalism to recast the alternating current and voltage as $I = \Re\mathrm{e}(I_0 e^{i\omega t})$, $V = \Re\mathrm{e}(V_0 e^{i\omega t})$, and identify the complex impedences of the resistor, inductor, capacitor to be respectively

$$Z_R = R \ ; \quad Z_L = i\omega L \ ; \quad Z_C = \frac{1}{i\omega C} \ . \tag{2.27}$$

Then we see that for a resistor the impedence is purely real, and Eq (2.23) simply reduces to Ohm's law. For inductors and capacitors the impedence is imaginary, and Eq (2.23) fully describes both the amplitude and the phase relationships between voltage and current. Impedences of elements connected in series or in parallel in an electric circuit are to be combined according to the same rules as resistances. We give examples below.

Example 2.17

The complex impedence of the RC series circuit



is given by the sum of the resistor and capacitor impedences in Eq (2.27),

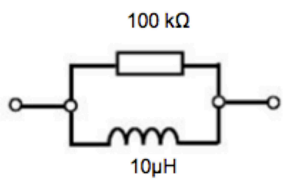
$$\begin{aligned} Z &= R + 1/(i\omega C) \\ &= \sqrt{R^2 + (\omega C)^{-2}}\, e^{-i\phi} \quad , \quad \tan\phi = 1/(\omega RC). \end{aligned} \tag{2.28}$$

Therefore, using Eq (2.23), the voltage lags the current by $\phi = \arctan(1/(\omega RC))$.

When, as shown in the figure, an ac voltage of amplitude V_1 is applied to the RC series circuit, the voltage drop across the resistor is given, by combining Eq (2.28) and Ohm's law, by $RI = R|V_1/Z| = V_1 R/\sqrt{R^2 + (\omega C)^{-2}}$. The voltage across the capacitor is given by $V_2 = |I(\omega C)^{-1}| = (\omega C)^{-1}|V_1/Z| = V_1/\sqrt{(\omega RC)^2 + 1}$.

Example 2.18

Consider the RL parallel circuit ($R = 100\ k\Omega$, $L = 10\ \mu H$)



The complex impedence is given by

$$\begin{aligned} Z^{-1} = R^{-1} + (i\omega L)^{-1} \ \Rightarrow \ Z &= i\omega RL/(R + i\omega L) \\ &= \omega RL e^{i\phi}/\sqrt{R^2 + \omega^2 L^2} \quad , \quad \tan\phi = R/(\omega L). \end{aligned} \tag{2.29}$$

Here the voltage leads the current by $\phi = \arctan(R/(\omega L))$.