University of Sheffield

PHY120 - Vectors

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1 Lecture 1

This lecture is an introduction to the concept of vectors and vector quantities. As a first step we will need to introduce some definitions needed to move our steps forwards. Being able to list and use these definitions is a fundamental prerequisite for lecture two. The main concepts presented here can be found at pages 367-398 of [1] (for further info also look at [2] and [3]). The aim of this part of the unit on vectors is for you to be able to:

- distinguish scalar and vector quantities
- multiply by a scalar, sum and subtract vectors geometrically
- draw a vector in the Cartesian plane and give the position vector of a point
- multiply by a scalar, sum and subtract vectors using their Cartesian form

1.1 Basic concepts of vectors

- Scalars vs Vectors. Some physical quantities can be fully mathematically described simply by a number indicating their magnitude (or size), and are called "*scalars*". For others, instead, more information is needed and both an intensity/magnitude and a direction have to be specified. These latter quantities are called "*vectors*" or "*vectorial*".

- Representations. Several ways can be used to represent them: arrows, stacks with a direction, or areas with a direction.

- Notations. A vector can be indicated in several different ways ($\overrightarrow{AB}$ or $\mathbf{a}$ or $\underline{a}$), and correspondingly its length or "*modulus*" is indicated as: $|\overrightarrow{AB}|$ or $|\mathbf{a}|$ or $|\underline{a}|$. A "*negative*" vector is a vector with same length of its corresponding positive but with opposite direction: the negative of $\overrightarrow{AB}$ is $\overrightarrow{BA}$.

- Multiplication by a scalar. A vector can be multiplied by a scalar. The effect of this multiplication is to "*stretch*" or "*contract*" its length (or modulus).

- Sum and subtraction. Two vectors are summed using the "*triangle law*". The subtraction of vectors is equal to the sum of one vector plus the "*negative*" of the second: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$. Vector addition is commutative [i.e. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$], associative [i.e. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$], and distributive with respect to a scalar [i.e. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$].

- Definitions. The "*resultant*" of two vectors **a** and **b** is their sum: $\mathbf{a}+\mathbf{b}$. A vector of unit length is indicated as $\hat{\mathbf{a}}$ and is found by dividing a vector by its modulus: $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$.

1.2 Cartesian components of vectors

- The unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Given a two-dimensional space, it is possible to define a Cartesian plane with axes x and y. The two vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the two unit vectors (or "*versors*") having length equal to 1 and pointing towards the positive x and positive y axes respectively.

- Vectors in the Cartesian plane. In order to write a vector in the Cartesian plane one can use the two unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. This can be understood using the addition law of vectors and the law of multiplication by a scalar. Any general vector $\overrightarrow{AB}$ in two dimensions can be written as: $k\hat{\mathbf{i}} + r\hat{\mathbf{j}}$. The two numbers k and r are called the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components of $\overrightarrow{AB}$, and can be written as a *row* or a *column*:

$$\text{(k, r)} \qquad \begin{pmatrix} k \\ r \end{pmatrix}.$$

- Basic vector operations in Cartesian plane. Using the representation of vectors in Cartesian coordinates it is possible to sum, subtract, and multiply a vector by a scalar. Given the two vectors $\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$ and $\mathbf{b} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}}$, and the scalar k one has:

$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\hat{\mathbf{i}} + (a_y + b_y)\hat{\mathbf{j}} \quad \text{and} \quad k\mathbf{a} = ka_x\hat{\mathbf{i}} + ka_y\hat{\mathbf{j}}$$

- Definition of position vector. The position vector of a point $P(x, y)$ with

Cartesian coordinates x and y is the vector $\mathbf{r} = \overrightarrow{OP} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

- Subtraction of position vectors. Consider two points $A(a_x, a_y)$ and $B(b_x, b_y)$ with position vectors: $\mathbf{a} = \overrightarrow{OA} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$ and $\mathbf{b} = \overrightarrow{OB} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}}$. Observing that $\overrightarrow{OA}+\overrightarrow{AB} = \overrightarrow{OB}$ it can be understood that a generic vector $\overrightarrow{AB}$ can be written as $\overrightarrow{OB} - \overrightarrow{OA} = \mathbf{a} - \mathbf{b} = (b_x - a_x)\hat{\mathbf{i}} + (b_y - a_y)\hat{\mathbf{j}}$.

- Modulus of a vector. Recalling that $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ form an angle of 90°, it is possible to calculate the modulus of a vector using Pythagoras' theorem: $|\mathbf{r}| = |x\hat{\mathbf{i}} + y\hat{\mathbf{j}}| = \sqrt{x^2 + y^2}$.

- Generalisation to N dimensions. The use of the Cartesian plane and of the Cartesian representation of vectors is extremely powerful and allows the generalisation to N dimensions, which would be extremely difficult to do by simply drawing vectors. For instance, in three dimensions one can define the unit vector (or "*versor*") $\hat{\mathbf{k}}$ aligned with the positive direction of z and a vector $\mathbf{a}$ can be written as $k\hat{\mathbf{i}} + r\hat{\mathbf{j}} + m\hat{\mathbf{k}}$, and represented by columns or rows with three numbers:

$$\text{(k, r, m)} \qquad \begin{pmatrix} k \\ r \\ m \end{pmatrix}.$$

Similarly its modulus will be $\sqrt{k^2 + r^2 + m^2}$. While the generalisation to three dimensions is still graphically possible, although more difficult than in two, the generalisation to 4, 5, ... N dimensions can be done only mathematically. For example given the two 5-dimensional vectors $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5)$: $\mathbf{c} + \mathbf{v} = (c_1 + v_1, c_2 + v_2, c_3 + v_3, c_4 + v_4, c_5 + v_5)$, and $|\mathbf{c}| = \sqrt{c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2}$.

2 Lecture 2

In this lecture we will introduce the concept of "*scalar product*", also called "*dot product*". After introducing its definition we will study its fundamental properties (distributivity, commutativity, etc...) and present several examples of calculations. The scalar and the vector products (the vector product is the subject of the next lecture) are the core of the vector-unit and constitute a fundamental calculation tool that will be useful in a vast range of modules in the Physics course. The subjects presented in this lecture correspond to pages 399 - 410 and 422 - 426 of [1] (for further info also look at [2] and [3]). The aim of this part of the unit on vectors is for you to be able to:

- define the scalar product of two vectors
- state and use the important properties of the dot product
- calculate the result of the dot product in vector and in Cartesian form
- use the dot product in some geometrical applications

2.1 The scalar or dot product

- Definition of the scalar product. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ it is possible to define the following scalar quantity: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|cos\theta$, where θ is the angle between the two vectors. This quantity is a scalar since it is the product of three scalar quantities: $|\mathbf{a}|$, $|\mathbf{b}|$, and $cos\theta$. This quantity is called "*scalar product*" of two vectors.

- Scalar product properties. For the scalar product the following properties holds: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutativity), $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributivity), and $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$. Moreover $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}|cos\theta = |\mathbf{a}|^2$, i.e. the scalar product of a vector times itself is equal to the square of its modulus.

- The scalar product of unit vectors. The easiest scalar product is the product among the unit vectors defining the three perpendicular axes of the Cartesian plane, since they have unitary length and they form and angle of either 90° or 0°:

$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=1 \qquad \hat{\mathbf{i}}\cdot\hat{\mathbf{j}}=\hat{\mathbf{i}}\cdot\hat{\mathbf{k}}=\hat{\mathbf{j}}\cdot\hat{\mathbf{k}}=0$$

- A more practical formula for the scalar product. Using these three properties and the result on the product among unitary vectors ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$), it is possible to derive an easy formula to calculate scalar products. Given two vectors $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}$ and $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}$, their scalar product is: $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2$. This can also be generalised to N-dimensions: $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3+a_4+b_4+...+a_Nb_N$.

- Angle between vectors. With the above formula it is possible to calculate the angle between two vectors:

$$cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

- Projection along a direction. If we calculate the scalar product between a vector $\mathbf{a}$ and a unitary vector $\hat{\mathbf{n}}$, we see that this is equivalent to the length of the projection of $\mathbf{a}$ along $\hat{\mathbf{n}}$: $\mathbf{a}\cdot\hat{\mathbf{n}}=|\mathbf{a}|cos\theta$. Equivalently we can also say that $\mathbf{a}\cdot\hat{\mathbf{n}}$ is the component of $\mathbf{a}$ in the direction of $\hat{\mathbf{n}}$.

- The vector equation of a line. In order to identify a line we need two points A and B, which can be associated to the two position vectors $\mathbf{a}$ and $\mathbf{b}$. A point P of coordinates (x,y,z) lays on the line passing through A and B if its position vector $\mathbf{r}$ is equal to the sum of $\mathbf{a}$ and a vector proportional to $\overrightarrow{AB}$, which can be written as: $\overrightarrow{AB}=k(\mathbf{b}-\mathbf{a})$. Therefore the vector equation of a line is: $\mathbf{r}=\mathbf{a}+k(\mathbf{b}-\mathbf{a})$. Alternatively one can also write this in "*vector column notation*" or using a "*Cartesian notation*", respectively:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+k\begin{pmatrix}b_1-a_1\\b_2-a_2\\b_3-a_3\end{pmatrix} \qquad \frac{x-a_1}{b_1-a_1}=\frac{y-a_2}{b_2-a_2}=\frac{z-a_3}{b_3-a_3}.$$

- Generalisation of Pythagoras' theorem. Considering the triangle ABC defined by the two vectors $\mathbf{b}=\overrightarrow{AB}$ and $\mathbf{c}=\overrightarrow{AC}$, the length of the vector $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$ is $|\mathbf{c}-\mathbf{b}|=\sqrt{(\mathbf{c}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})}=\sqrt{|\mathbf{c}|^2+|\mathbf{b}|^2-2|\mathbf{c}||\mathbf{b}|cos\theta}$. If the two vectors $\mathbf{b}$ and $\mathbf{c}$ form an angle of 90° the above equation reduces to Pythagoras' theorem, since $cos\theta=0$.

3 Lecture 3

In this lecture we will introduce the concept of "*vector product*", also called "*cross product*". After introducing its definition we will study its fundamental properties and present several examples of calculations. The vector product, together with the scalar product presented in Lecture 2 are the core of the vector-unity and constitute a fundamental calculation tool that will be useful in a vast range of modules in the Physics course. The subjects presented in this lecture correspond to pages 411 - 421 and 427 - 431 of [1] (for further info also look at [2] and [3]). The aim of this part of the unit on vectors is for you to be able to:

- define the vector product of two vectors
- state and use the important properties of the cross product
- calculate the result of the cross product in vector and in Cartesian form
- use the vector product in some geometrical applications

3.1 The vector or cross product

- Definition of the vector product. Two general vectors in three dimensions $\mathbf{a}$ and $\mathbf{b}$ with a common origin identify a plane. Given these two vectors it is possible define the quantity: $\mathbf{a} \times \mathbf{b} = \hat{\mathbf{e}}|\mathbf{a}||\mathbf{b}|sin\theta$, where θ is the angle formed between the two vectors and $\hat{\mathbf{e}}$ is the vector perpendicular to the plane formed by $\mathbf{a}$ and $\mathbf{b}$ in the direction set by the "*right-handed screw rule*" or "*right-hand rule*" (see below). This quantity is a vector because $\hat{\mathbf{e}}$ is a vector and $|\mathbf{a}|$, $|\mathbf{b}|$, and $sin\theta$ are scalars.

- The right-hand rule. Consider two vectors in three dimensions $\mathbf{a}$ and $\mathbf{b}$ having a common origin and the plane containing them. Now lets put the right thumb perpendicular to the plane choosing its direction in order to have the other fingers of the right hand going from $\mathbf{a}$ to $\mathbf{b}$. The direction set by the right thumb in these conditions is the direction of the unitary vector $\hat{\mathbf{e}}$ needed to calculate the vector product $\mathbf{a} \times \mathbf{b}$.

- Anti-commutativity. Considering the same two vectors $\mathbf{a}$ and $\mathbf{b}$ as before it is possible now to calculate $\mathbf{b} \times \mathbf{a}$ (not $\mathbf{a} \times \mathbf{b}$). To identify the unit vector $\hat{\mathbf{e}}$ now one needs to set the fingers of the right-hand to go from $\mathbf{b}$ to $\mathbf{a}$, therefore reversing the direction of the right thumb and consequently the direction of $\hat{\mathbf{e}}$. This shows that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, i.e. the vector product is anti-commutative.

- Simple cases. The most simple case of vector product is the product of a unit vector times itself: $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}|sin\theta = 0$, since $\theta = 0°$, and similarly: $\hat{\mathbf{j}} \times \hat{\mathbf{j}} = 0°$ and $\hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0°$.

- Vector product of the Cartesian unit vectors. Consider now the three unit vectors forming the three dimensional Cartesian space: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. The six following relations can be calculated:

$$\begin{array}{ccc} \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} & \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} & \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \\ \hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}} & \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} & \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}. \end{array}$$

- Distributivity and multiplication by a scalar. The distributivity relation: $\hat{\mathbf{a}} \times (\hat{\mathbf{b}} + \hat{\mathbf{c}}) = \hat{\mathbf{a}} \times \hat{\mathbf{b}} + \hat{\mathbf{a}} \times \hat{\mathbf{b}}$. Here we only demonstrate that it holds for the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$: $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} + \hat{\mathbf{k}}) = \hat{\mathbf{i}} \times \hat{\mathbf{j}} + \hat{\mathbf{i}} \times \hat{\mathbf{k}}$. Moreover the following relation holds for the multiplication by a scalar: $k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$.

- The vector product in Cartesian components. Using the relations found for the vector product of two unit vectors, the distributivity and multiplication by a scalar relations, and the fact that a vector can always be decomposed using unit vectors, it is possible to obtain the following formula to express the vector product:

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\hat{\mathbf{i}} - (a_1 b_3 - a_3 b_1)\hat{\mathbf{j}} + (a_1 b_2 - a_2 b_3)\hat{\mathbf{j}}.$$

- A mnemonic technique. In order to remember the above expression it is possible to use the following mnemonic technique. It will be seen in the module/unit on matrices that this technique is not just mnemonic but corresponds to the determinant of a square 3×3 matrix. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ their vector product can be found by arranging them in a square and by multiplying and subtracting elements along partial diagonals:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

- The area of a triangle. Given a triangle of vertices ABC and of length side a, b, and c, its area is equal to $\frac{1}{2}bcsin\theta$. This is equal to half the modulus of the vector product $|\overrightarrow{AB} \times \overrightarrow{AC}|/2 = |\overrightarrow{AB}||\overrightarrow{AC}||sin\theta|/2$.

- The vector equation of a plane A set of three points A, B, and C in a three dimensional space identify a plane and the vector $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to it. In order to give the equation of the plane one can observe that for any point P belonging to it, the vector $\overrightarrow{AP}$ will be perpendicular to $\mathbf{n}$. Therefore if we indicate with $\mathbf{a}$ the position vector of A and with $\mathbf{r}$ the position vector of P the following relation holds: $\mathbf{n} \cdot \overrightarrow{AP} = \mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$. Given three points A, B, and C the vector equation of the plane containing them is therefore: $(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot (\mathbf{r} - \mathbf{a}) = 0$. A simpler case is when one has to look for the plane perpendicular to a direction $\hat{\mathbf{n}}$ and passing by a point A, with position vector $\mathbf{a}$. In that case the vector equation of the plane reduces to $\hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{a}) = 0$, which can be rearranged as:

$$\hat{\mathbf{n}} \cdot \mathbf{r} = \hat{\mathbf{n}} \cdot \mathbf{a} \quad .$$

References

[1] Mathematics for Physicists and Astronomers Volume 1, compiled by David Mowbray, PEARSON Education Limited (2015).

[2] Hyper Physics webpage (http://hyperphysics.phy-astr.gsu.edu/hphys.html) on the Georgia State University website (USA).

[3] Unit on vectors of the “Multivariable Calculus” course of the MIT Open Courseware (http://ocw.mit.edu/courses/mathematics/18-02-multivariable-calculus-fall-2007/index.htm) on the MIT website (USA).