PHY120 - Unit 1: Functions and Differentiation

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1 Functions: definition, examples, graphs

A 'function' $y = f(x)$ is a rule that associates a particular value y in a set B with each value x in a set A; x is called an argument of a function $y = f(x)$. In other words, if an independent variable x determines the value of a dependent variable y, then y is a function of x (if a numerical value of x is given, a single value of y is determined). Another definition: A function is a rule that operates on an input to produce a single output from this input.

In physics most (but not all) functions associate real numbers with real numbers. Examples:

- Position and velocity of an object are associated with time.
- Electric current is associated with the applied voltage and resistance.
- Kinetic energy of the molecules is associated with the temperature.

Functions $y = f(x)$ can always be presented as graphs.

Simple functions and their geometrical interpretation:

1. The volume of a rectangular parallelepiped is calculated as

$$V = x \times y \times z \tag{1}$$

   where x, y and z and the length, width and height of the parallelepiped, or its dimensions. If $x = y = z$, then this is a cube and its volume is given by:

$$V = x^3 \quad \text{or} \quad y = x^3 \tag{2}$$

   This can be presented as a graph where V depends on x.

   In this case V is a function of x. Once x is given the associated value of V is determined. The set A consists of real numbers $x \geq 0$.

2. The velocity of an object falling down from the top of the building is expressed by

$$v = v_0 + g \times t \tag{3}$$

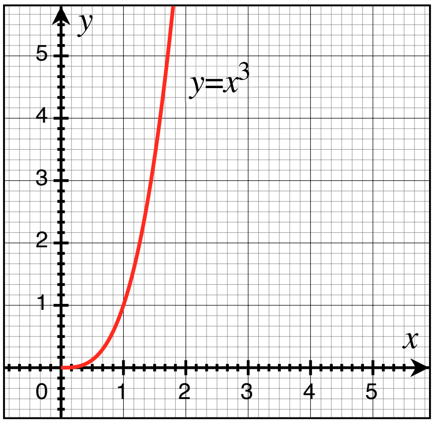

Figure 1: Volume of a cube as a function of its side length

where v_0 is the initial velocity, $g = 9.81$ m/s^2 is the acceleration due to gravity (also known as free fall acceleration) and t is the time. (Here we neglected the resistence of the air). In general case the velocity v can be determined for any value of time t. In reality, however, the velocity obeys this expression only when the distance of the object from the surface of the Earth is more than 0. So, unless we know the initial height of the building we cannot find out the set A of values t for which the function is valid.

3. Sometimes the function $f(x)$ is not known, although it is known that the dependence between y and x exists. For example, the temperature T at any particular place on the Earth is a function of time t but the exact formula (dependence) is not known.

A function associates with each value x from some set A exactly **one** value y from another set B. We say that f defines y as a function of x and write $y = f(x)$. **The set A is called the domain of a function f. The set B is called the range of f – the set of all numbers $y = f(x)$ obtained by allowing x to vary over the domain of f.**

In physics in most (but not all) cases we deal with real numbers and real functions.

2 Basic functions

2.1 Polynomial functions

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n \tag{4}$$

Examples:

- $y = 2 \quad (a_0 = 2)$
- $y = 2 + x \quad (a_0 = 2, a_1 = 1)$

  This is similar to the example above of the free falling object but with different coefficients.
- $y = 2 + x - 3x^2 \quad (a_0 = 2, a_1 = 1, a_2 = -3)$

The domain of these functions includes all real numbers.

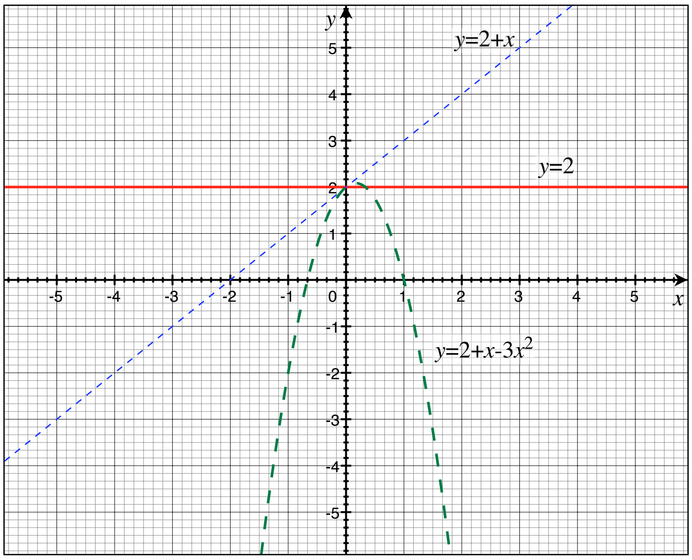

Figure 2: Examples of polynomial functions

2.2 Rational functions

A general rational function has the form:

$$f(x) = \frac{p(x)}{q(x)} \tag{5}$$

where both functions $p(x)$ and $q(x)$ are polynomials. The domain of such a function is all real numbers except those values of x for which the denominator $q(x) = 0$.

2.3 Trigonometric functions

$$y = \sin\theta; \quad y = \cos\theta; \quad y = \tan\theta = \frac{\sin\theta}{\cos\theta}; \quad y = \cot\theta = \frac{1}{\tan\theta} \tag{6}$$

The sign of the trigonometric functions depends on the quadrant where the angle θ lies (Figures 3, 4).

The domains of sine and cosine functions are all real numbers, their ranges are $[-1, 1]$ – closed interval meaning $-1 \leq y \leq 1$ (example of an open interval (-1,1): $-1 < y < 1$).

The tangent function is not defined for values of θ which are odd multiples of $\pi/2$ (we can say $\theta \neq (2n-1)\pi/2$, where n is an integer number). So the domain of the tangent function is all real numbers except $\theta = (2n-1)\pi/2$, where n is an integer number. The range of tangent function is all real numbers.

[Write down the domain and the range of the function $y = \cot x$]

Figure 5 shows the graphs of trigonometric functions. On this graph the argument of the trigonometric functions is denoted by a letter x, instead of the greek letter θ as on Figure 3.

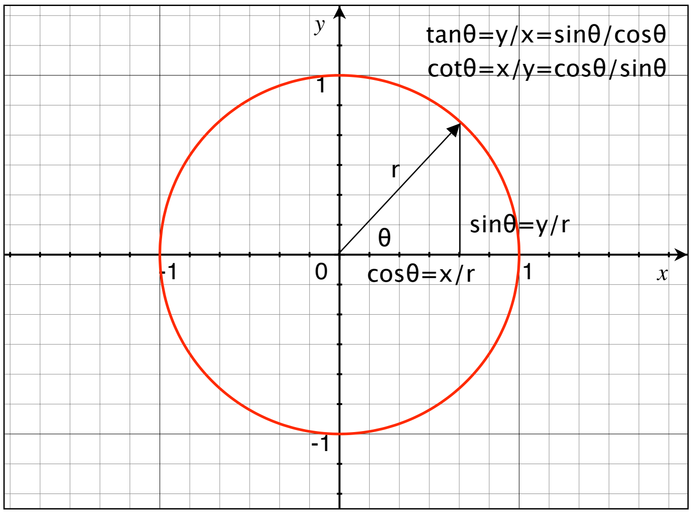

Figure 3: Definition of trigonometric functions

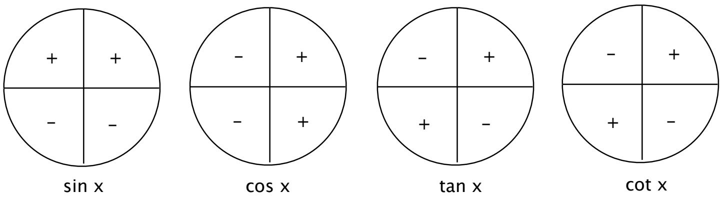

Figure 4: Signs of trigonometric functions

Some useful relations:

$\sin x = 0$ if $x = n\pi$; $\cos x = 0$ if $x = (2n-1)\pi/2$

$\cos x = (-1)^n$ if $x = n\pi$; $\sin x = (-1)^{n+1}$ if $x = (2n-1)\pi/2$

where n is an integer number.

Reflection relative to the x-axis (Figure 3) does not change x-coordinate, but changes the sign of the y-coordinate that results in the transformation of the angle θ into $-\theta$, therefore:

$\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$; $\cot(-\theta) = ?$

Cosine is an **even** function, sine and tangent are **odd** functions, cot –?

Rotating the radius (Figure 3) anticlockwise through $\pi/2$, π or 2π gives the following relations:

$\sin(\theta + \pi/2) = \cos\theta$; $\sin(\theta + \pi) = -\sin\theta$; $\sin(\theta + 2\pi) = \sin\theta$;

$\cos(\theta + \pi/2) = -\sin\theta$; $\cos(\theta + \pi) = -\cos\theta$; $\cos(\theta + 2\pi) = \cos\theta$;

$\tan(\theta + \pi/2) = -\cot\theta$; $\tan(\theta + \pi) = \tan\theta$; $\tan(\theta + 2\pi) = \tan\theta$;

$[\cot(\theta + \pi/2) = ?$ $\cot(\theta + \pi) = ?$ $\cot(\theta + 2\pi) = ?]$

Tangent function repeats after an interval of π (has a period of π), whereas sine and cosine functions

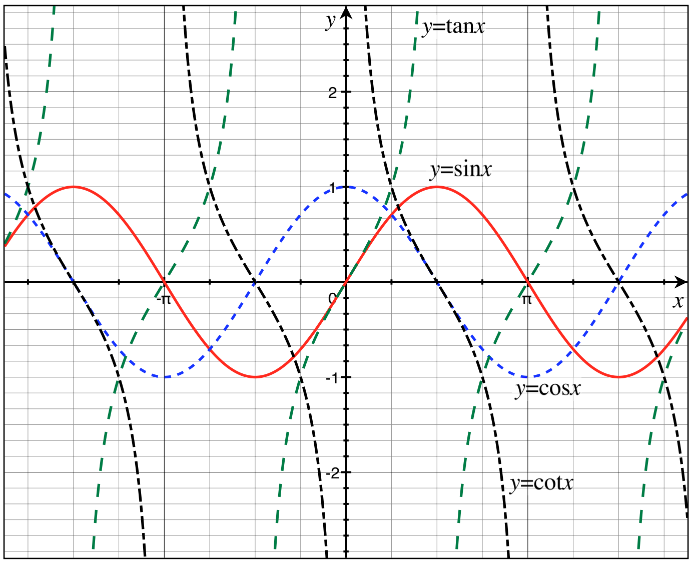

Figure 5: Trigonometric functions

have a period of 2π. The graph of the cosine function is obtained from the graph of the sine function by shifting it by $\pi/2$ to the left.

Exercises:

1. Using Figure 3 show that $\cos^2\theta + \sin^2\theta = 1$.

Other relations between trigonometric functions:

Sum formulae:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \tag{7}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \tag{8}$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \tag{9}$$

Double angle formulae

$$\sin 2x = 2 \sin x \cos x \tag{10}$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \tag{11}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} \tag{12}$$

From the double angle formulae we can get:

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{13}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{14}$$

which will be useful for integration.

From the sum formulae we can obtain:

$$\sin(a + b) - \sin(a - b) = 2 \cos a \sin b \tag{15}$$

and if we make a substitution: $x = a + b$ and $y = a - b$, then $a = \frac{1}{2}(x + y)$ and $b = \frac{1}{2}(x - y)$, and we can also write:

$$\sin x - \sin y = 2 \sin \left[\frac{1}{2}(x - y)\right] \cos \left[\frac{1}{2}(x + y)\right] \tag{16}$$

$$\sin x + \sin y = 2 \sin \left[\frac{1}{2}(x + y)\right] \cos \left[\frac{1}{2}(x - y)\right] \tag{17}$$

Similarly,

$$\cos x - \cos y = -2 \sin \left[\frac{1}{2}(x - y)\right] \sin \left[\frac{1}{2}(x + y)\right] \tag{18}$$

$$\cos x + \cos y = 2 \cos \left[\frac{1}{2}(x + y)\right] \cos \left[\frac{1}{2}(x - y)\right] \tag{19}$$

It is useful to know also the *secant* and *cosecant* functions (similar to the *cotangent* function: $\cot x = 1/\tan x$):

$\sec x = \frac{1}{\cos x}$; $\mathrm{cosec}\ x = \frac{1}{\sin x}$;

Note that the third letters of the sec, cosec and cot functions, *c*, *s* and *t*, are the first letters of the original functions cos, sin and tan.

There are a few other rules relevant to triangles:

- Cosine rule. In a triangle with side lengths a, b, c and opposite angles A, B, C,

$$c^2 = a^2 + b^2 - 2ab \cos C \tag{20}$$

  Similarly,

$$a^2 = b^2 + c^2 - 2bc \cos A \tag{21}$$

- Sine rule. In a triangle with side lengths a, b, c and opposite angles A, B, C,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{22}$$

- Area of a triangle. In a triangle with side lengths a, b, c and opposite angles A, B, C, the area S is calculated as

$$S = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B \tag{23}$$

Exercises:

1. Write down the values for the following functions **without using the calculator**:

a) $\cos \pi/6$, $\cos \pi/4$, $\cos \pi/3$.

b) $\sin \pi/6$, $\sin \pi/4$, $\sin \pi/3$.

c) $\tan \pi/6$, $\tan \pi/4$, $\tan \pi/3$.

d) $\cos 2\pi/3$, $\cos 3\pi/4$, $\cos 5\pi/6$.

e) $\sin 2\pi/3$, $\sin 3\pi/4$, $\sin 5\pi/6$.

d) $\cot 2\pi/3$, $\cot 3\pi/4$, $\cot 5\pi/6$.

2. Use the formulae given above to deduce that $\sec^2 x = 1 + \tan^2 x$.

See also Recommended textbook, Chapter 9, Pages 47-110, exercises on pages 52, 57, 58, 68-69, 73, 85, 93.

2.4 Harmonic function

If there is a function $a \cos x + b \sin x$, where x is an argument and a and b are the constants, then this function can be put in the form $c \cos(x + \phi)$, where constants c, and ϕ can be obtained from a and b. The numbers a and $-b$ can always be presented as coordinates of the point $(a, -b)$. The distance from the origin of the coordinate system to this point (or the polar coordinate of this point) is: $c = \sqrt{a^2 + b^2}$ and the polar angle ϕ can be calculated as: $\cos \phi = a/c$ and $\sin \phi = -b/c$. So we can write

$$a \cos x + b \sin x = c(\frac{a}{c} \cos x + \frac{b}{c} \sin x) = c(\cos \phi \cos x - \sin \phi \sin x) = c \cos(x + \phi) \qquad (24)$$

A function of the form $c \cos(wx + \phi)$ is called a harmonic function or a sinusoid in the variable x. Obviously, sin and cos can be changed one into another by shifting the function by $\pi/2$. The function is like a cosine function stretched or compressed horizontally to an extent proportional to w and translated (or shifted to) a distance $-\phi/w$ along the x-axis. The function has a period of $2\pi/w$. Such functions describe harmonic motion and wave-like time or space variations. Examples include harmonic motion of a pendulum, alternating electric current etc.

See also Recommended textbook, Chapter 9, Pages 95-107, exercises on pages 97, 107.

2.5 Exponential function

$$y = a^x \qquad (25)$$

The domain of the function is $a > 0$ and x is real. The range is real numbers $(0, \infty)$. This is the most general case. In some specific cases we can have, for example, $a = -3$ and $x = 3$. so $y = (-3)^3 = -27$, which is a valid result. However, here and hereafter we will use the general definition of the exponential function with the domain $a > 0$ and real x.

Index laws:

$$a^x \times a^y = a^{x+y}; \qquad \frac{a^x}{a^y} = a^{x-y}; \qquad (a^x)^y = a^{xy}; \qquad a^0 = 1; \qquad (26)$$

Exercise:

1. Sketch the graph 0.5^x.

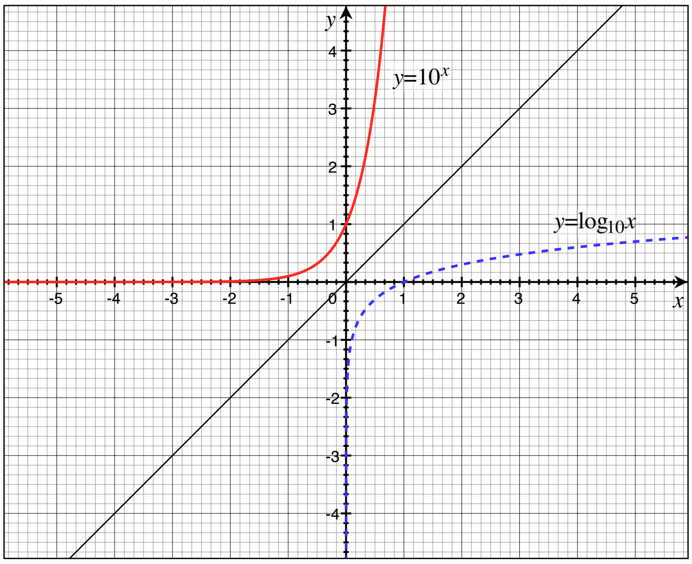

Figure 6: Exponential and logarithmic functions

2.6 Logarithmic function

If $a > 0$ ($a \neq 1$), then for each $x > 0$ there is a unique y such that $a^y = x$. The number y **is called the logarithm to base** a **of** x and is denoted as:

$$y = \log_a x \tag{27}$$

$y = \log_a x$ if and only if $a^y = x$.

The graph $y = \log_a x$ is obtained from that of $y = a^x$ by reflection with respect to the line $y = x$ (Figure 6).

The domain of the logarithmic function is $x = (0, \infty)$ and the range is all real numbers. $\log_a 1 = 0$ for $a > 0$.

The exponential function a^x and the logarithmic function $\log_a x$ are inverse of one another in the sense that each has the effect of undoing what the other does:

$\log_a(a^x) = x$ (x is a real number); $a^{\log_a x} = x(x > 0)$;

The rules for the exponential functions determine the rules for the logarithmic functions (remember that $y = \log_a x$ if $a^y = x$).

$$\log_a x + \log_a y = \log_a(xy); \quad \log_a x - \log_a y = \log_a\left(\frac{x}{y}\right); \quad \log_a 1 = 0; \quad \log_a(x^b) = b\log_a x \tag{28}$$

2.7 Function e^x

There is a special value of the base a for which differentiation of both a^x and $\log_a x$ gives quite simple results (details later). This base is denoted by e and can be defined by the formula:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = 1 + 1 + \frac{1}{2} + \frac{1}{6} + ... \tag{29}$$

A simple calculation shows that $e \approx 2.71828$.

In general, for any real number x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... \tag{30}$$

from which the formula for e can be obtained by putting $x = 1$. The right-hand side of the Eq. (30) is called a **series** (more on expansion series later). If the sum is calculated for large values of n, then it will be close to e^x.

$\log_e x$ is written simply as $\ln x$ and is called natural logarithm of x. Since $x = a^{\log_a x}$, we can deduce that $\ln x = \ln(a^{\log_a x}) = (\log_a x) \times (\ln a)$ and hence:

$$\log_a x = \frac{\ln x}{\ln a} \tag{31}$$

Similar relation is valid for logarithm to a base different from e. In the general case

$$\log_a x = \frac{\log_b x}{\log_b a} \tag{32}$$

Exercise:

1. Simplify the equation: $y = e^{x \ln x}$.

Physics examples of the exponential functions include:

- Radioactive decay. The number of radioactive atoms decreases with time according to the exponential law $N(t) = N_0 e^{-t/t_0}$, where N_0 is the initial number of radioactive atoms and t_0 is the life-time of the radioactive isotope.
- The X-radiation or γ-radiation is absorbed in a material following an expression $I(x) = I_0 e^{-x/x_0}$, where I_0 and $I(x)$ are the initial intensity of the radiation and the intensity of the radiation after traveling the distance x in the material, respectively, x_0 is the coefficient which depends on the material and on the wavelength (energy) of the radiation.
- The charged capacitor in an electric circuit is discharged following an exponential $I(t) = \frac{V_0}{R} e^{-\frac{t}{RC}}$, where V_0 is the initial voltage across the capacitor, I is the current after the time t, R is the resistance and C is the capacitance.

See also Recommended textbook, Chapter 8, Pages 1-46, exercises on pages 5, 7, 10, 19, 20, 24, 31-32, 44-45.

2.8 Hyperbolic functions

The hyperbolic functions sinh, cosh, tanh and coth are closely related to the exponential function e^x and are named by analogy with trigonometric functions:

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right); \;\; \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right); \;\; \tanh x = \frac{\sinh x}{\cosh x}; \;\; \coth x = \frac{\cosh x}{\sinh x}; \tag{33}$$

The domains of these functions are real numbers and their ranges are: $\sinh x$ – $(-\infty, \infty)$, cosh – $[1, \infty)$, $\tanh x$ – $(-1, 1)$ (see Figure 7). There are also corresponding reciprocals: sech and cosech (coth is also a reciprocal of tanh).

$\cosh(-x) = \cosh x$ is an even function, whereas $\sinh(-x) = -\sinh x$ and $\tanh(-x) = -\tanh x$ are odd functions.

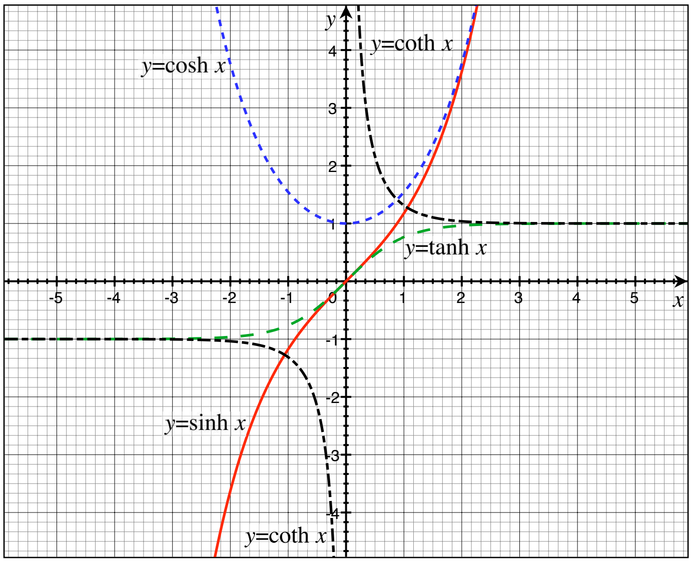

Figure 7: Hyperbolic functions

Exercise:

1. Use the definitions of the hyperbolic functions to show that:

$\cosh^2 x - \sinh^2 x = 1$;

$\sinh 2x = 2 \sinh x \cosh x$.

Hyperbolic functions satisfy addition and double angle formulae similar to those for the trigonometric functions (note that + and − signs are sometimes different from those for trigonometric relations):

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{34}$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{35}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x \tag{36}$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1) \tag{37}$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1) \tag{38}$$

See also Recommended textbook, Chapter 8, Pages 11-16, exercises on page 15.

3 Inverse functions

If there is a function $y = f(x)$, then (in the simplest case) different values of x give different values of y. Then for each y from the range of f there is precisely one x in the domain of f for which $y = f(x)$. So we can also define an inverse function $x = f^{-1}(y)$. **It is essential not to confuse $f^{-1}(y)$ with 1/f(y); the former is the inverse and the latter is the reciprocal.** The inverse function $f^{-1}(y)$ means that we solve the equation $y = f(x)$ to find x as a function of y.

Consider simple examples:

$y = x^3$. Solving this equation we find $x = \sqrt[3]{y}$, so $f^{-1} = \sqrt[3]{y} = y^{1/3}$.

If instead we took $y = \sqrt[3]{x}$, then $x = f^{-1} = y^3$.

In general case: the inverse function of f^{-1} is f, i. e. f and f^{-1} are the inverse functions to each other.

Another example is the one considered earlier – exponential and logarithmic functions are inverse to each other: $x = \log_a y$ if and only if $y = a^x$.

The graph of the inverse function is the graph of the original function reflected in the line $y = x$ (see Figure 6).

If $y = x^2$, where the domain is all real numbers, then for each value of y there are two values of x: $x = \pm\sqrt{y}$, so two separate functions $x = \sqrt{y}$ and $x = -\sqrt{y}$. By definition only positive values of x are considered as an inverse function of $y = x^2$, so the inverse function will be $x = \sqrt{y}$. Note that the solution of the equation: $y = x^2$, will still be two functions: $x = \pm\sqrt{y}$, but only one of them (positive) is defined as an inverse of $y = x^2$.

3.1 Inverse trigonometric functions

Like in the function $y = x^2$, in the trigonometric functions a certain value of y corresponds to multiple values of an argument x: $f(x) = f(x + 2\pi)$.

The range of $\cos x$ is [-1,1]. If we assume that $0 \leq x \leq \pi$, then the function $\cos x$ (see Figure 5) decreases from 1 to -1 taking each value precisely once. So for each value of y there is precisely one value of x. In this simple case we can define an inverse function $\cos^{-1}$ or arccos: $x = \arccos y$ for $0 \leq x \leq \pi$. The domain of arccos is [-1,1] and its range is [0,π].

Similar (but not the same) restrictions can be applied to the domains of the functions sin and tan to make the inverse functions simpler. The domains of the functions $y = \sin x$ and $y = \tan x$ can be restricted to the intervals $[-\pi/2, \pi/2]$ and $(-\pi/2, \pi/2)$, respectively and the inverse functions defined as: $x = \arcsin y$ for $-\pi/2 \leq x \leq \pi/2$ and $x = \arctan y$ for $-\pi/2 < x < \pi/2$.

Example

Solve the equation and write the answer as a fraction of π

$\sin x = -\frac{\sqrt{3}}{2}$;

$\sin(-x) = -\sin x$; $\sin(-x) = \frac{\sqrt{3}}{2}$;

You should know that to have $\sin x = \frac{\sqrt{3}}{2}$ the argument x should be equal to $x = \frac{\pi}{3}$ (see exercises to the section on trigonometric functions). In our case $-x = \frac{\pi}{3}$, so $x = -\frac{\pi}{3}$. In terms of the inverse function we can write:

$y = \sin x = -\frac{\sqrt{3}}{2}$; $x = \arcsin y = \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

In fact we know that sine is a periodic function and the exact answer is $x = \arcsin\left(-\frac{\sqrt{3}}{2}\right) + 2\pi n$, and $x = -\pi + \arcsin\left(\frac{\sqrt{3}}{2}\right) + 2\pi n$, where n is an integer. This gives the combined solution $x = -\frac{\pi}{2} \pm \frac{\pi}{6} + 2\pi n$. However, in some cases we may need to find solutions within a certain range of the argument x, for instance, for x only within the range of $\arcsin y$: $-\pi/2 \leq x \leq \pi/2$. In this case, we have only one solution: $x = -\frac{\pi}{3}$.

3.2 Inverse hyperbolic functions

The function $y = \cosh x$ has the domain of all real numbers and the range $[1,\infty)$ (see Figure 7). It is similar to $y = x^2$ in that it is even $\cosh(-x) = \cosh x$. Restricting the domain of the function to the interval $[0,\infty)$, we can define the inverse function $x = \cosh^{-1} y$ for $x \geq 0$ (also denoted arcosh by analogy with inverse trigonometric functions). The domain of $\cosh^{-1}$ is $[1,\infty)$ and its range is $[0,\infty)$.

Similarly we can define the inverse of the functions sinh and tanh: $x = \sinh^{-1} y$=arsinh y and $x = \tanh^{-1} y$=artanh y.

Exercises: 1. Find the domains and the ranges of the functions arsinh and artanh.

4 Composite functions

Suppose there is a function $y = g(x)$. Suppose there is also a function $z = f(y)$. Then we can define a composite function $z = f(y) = h(x)$ which is usually denoted as:

$$h(x) = f(g(x)) \tag{39}$$

Let us consider an example. Suppose the function $y = g(x) = x^2$ and the function $z = f(y) = \sin y$. Then the composite function $h(x) = \sin(x^2)$. We can also construct the composite function $t(y) = \sin^2 y$.

5 Series and limits

A *sequence* is a set of numbers written in a specific order.

Examples: $1, 3, 5, 7, 9$ and $-1, -2, -3, -4, -5, ...$

Each number in a sequence is call a *term* of the sequence.

Sequences can be finite (the first one) and infinite (the 2nd one).

The sequence of odd numbers can be denoted as: $x[k] = 2k - 1$, where $k = 1, 2, 3, 4, 5$ (can also be extended indefinitely).

Consider the sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$ that can be denoted as $x[k] = \frac{1}{k}$. It is clear that when k tends to infinity, $x[k]$ tends to 0. This can be written in terms of a limit:

$\lim_{k \to \infty} x[k] = 0.$

A sequence is said to *converge* if it possesses a finite limit as k tends to infinity. When a sequence does not possess a finite limit as k tends to infinity, it is said to *diverge*. The infinite sequence of odd numbers is diverging because it does not possess a finite limit.

A *series* is the sum of all terms in a given sequence. For instance the sum of 5 odd numbers (see example above) is a series and can be written as

$\sum_{k=1}^{5} (2k - 1)$

Infinite series are the series where the terms of an infinite sequence are added together. The series $\sum_{k=1}^{\infty} (2k - 1)$ are the infinite series. Infinite series can converge (the sum of all terms of a sequence is finite) or diverge (the sum is infinite). This can be determined using certain convergence criteria (not covered here). The convergence can be determined using *the sequence of partial sums*. Consider the infinite series formed from the sequence $x[k] = \frac{1}{2^k}$, where $k = 0, 1, 2, ...$: $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

We can calculate the sum of n terms of this series, $S_n = \sum_{k=1}^{n} \frac{1}{2^k}$, for different values of n:

$S_1 = 1, S_2 = 1 + \frac{1}{2} = 1.5, S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1.75.$

The sequence S_n is called the sequence of partial sums. You can see that the difference between the two consecutive terms of this sequence, $S_{n+1} - S_n$, decreases as n increases, leading to the conclusion that the sequence of partial sums or series S_n should converge. In fact this series converges to 2: $\sum_{k=1}^{\infty} \frac{1}{2^k} = 2.$

For any infinite series, $\sum_{k=1}^{\infty} x[k]$, the sequence of partial sums can be written as: $S_1 = x[1], S_2 = x[1] + x[2], S_3 = x[1] + x[2] + x[3], ...$ If the sequence S_n converges to a limit S, then the infinite series has sum S or converges to S.

An *arithmetic sequence* is a sequence of numbers where each new term after the first one is formed by adding a fixed amount (*common difference*) to the previous term in the sequence. The sequence of odd numbers is an arithmetic sequence with the common difference 2.

In general, an arithmetic sequence can be defined as: $a, a + d, a + 2d, ..., a + (n - 1)d$, where d is the common difference and $n = 1, 2, 3, 4, ...$ The *arithmetic series* or the sum of the n terms of an arithmetic sequence is given by:

$$S_n = \frac{n}{2} [2a + (n - 1)d] \tag{40}$$

A *geometric sequence* is a sequence of numbers where each term after the first one is found by multiplying the previous term by a fixed number (*common ratio*). For example, the sequence $1, 3, 9, 27, ...$ is a geometric sequence with the first term 1 and common ratio 3.

In general, a geometric sequence can be defined as: $a, ar, ar^2, ..., ar^{n-1}$, where r is the common ratio and $n = 1, 2, 3, ...$

The *geometric series* or the sum of the n terms of a geometric sequence is given by:

$$S_n = \frac{a(1 - r^n)}{1 - r} \tag{41}$$

(valid only if $r \neq 1$)

If the common ratio in a geometric sequence is less than 1 in modulus (i.e. $-1 < r < 1$), then the geometric series or the sum of the infinite number of terms converges and can be calculated (sum to infinity S_∞).

The *binomial theorem* gives a polynomial expansion to the sum of two variables (a and b are any real numbers) to the power of n (n is a positive integer):

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \\ &+ \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + ... + \frac{n!}{k!(n-k)!}a^{n-k}b^k + ... + b^n\end{aligned} \tag{42}$$

The *binomial theorem* is often used in a particular case when $a = 1$ an $b = x$:

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \\ &+ \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + ... + \frac{n!}{k!(n-k)!}x^k + ... + x^n\end{aligned} \tag{43}$$

If the power index is not a positive integer then we get an infinite *binomial series* (valid for $-1 < x < 1$ and any a):

$$\begin{aligned}(1+x)^a &= 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 \\ &+ \frac{a(a-1)(a-2)(a-3)}{4!}x^4 + ... + \frac{\prod_{k=1}^{n}(a-k+1)}{n!}x^n + ...\end{aligned} \tag{44}$$

where k and n are integer numbers: $k \geq 1$ and $n \geq 1$. (Note that the general formula for the n-term of the series is invalid for the first term $n = 0$).

See also Recommended textbook, Chapter 19, Pages 629-648, exercises on pages 630, 633, 634, 636, 638, 646, 648.

6 Differentiation: definition

We introduce differentiation by considering the speed of an object (for example a car or a molecule) and the gradient of a curve.

Suppose an object is moving along a straight line and its distance s from a fixed point on the line at time t is given by a function $s = f(t)$. Let us calculate the speed of this object. Consider the time interval $[t, t + \delta t]$, where δt is a small change in time. Let the corresponding change in the distance s be δs. At the beginning of the time interval (at time t) the distance was $s = f(t)$, whereas at the end of this time interval (at time $t + \delta t$) the distance became $s = f(t + \delta t)$. Thus the average speed of an object over the small time interval is

$$\frac{\text{distance travelled}}{\text{time taken}} = \frac{\delta s}{\delta t} = \frac{f(t+\delta t) - f(t)}{\delta t} \tag{45}$$

The speed v of an object at time t is the limit of this average speed as δt tends to 0:

$$v = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t} \tag{46}$$

Example:

An object is dropped from the height of 20 meters. Its distance from the surface is described by the formula: $s = f(t) = 20 - 4.9t^2$ (we neglect here the air resistance). Let us find the speed of the object as a function of time.

$$\delta s = f(t + \delta t) - f(t) = 20 - 4.9(t + \delta t)^2 - (20 - 4.9t^2) = -9.8t\delta t - 4.9(\delta t)^2 \tag{47}$$

$$v = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \lim_{\delta t \to 0} \frac{-9.8t\delta t - 4.9(\delta t)^2}{\delta t} = \lim_{\delta t \to 0} (-9.8t - 4.9(\delta t)) = -9.8t \tag{48}$$

The minus sign indicates that the object is traveling in the direction of s decreasing.

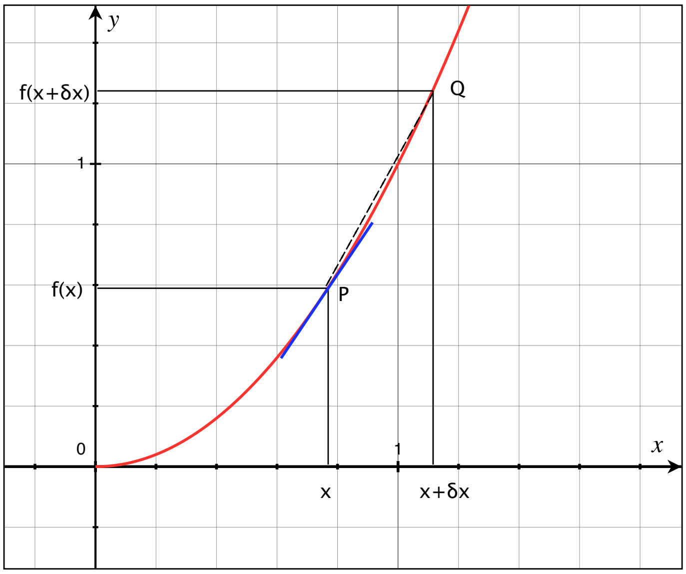

Figure 8: Illustration of the gradient to the function

Let us now consider the *gradient* or slope of the tangent to the graph (see Figure 8) of a function $y = f(x)$ at a certain point $P(x, y)$. We choose first the nearby point on the graph $Q(x+\delta x, y+\delta y)$. Then $y = f(x)$, $y + \delta y = f(x + \delta x)$, so $\delta y = f(x + \delta x) - f(x)$. The slope of the chord (or the segment) PQ is given by:

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} \tag{49}$$

With decreasing δx the slope of PQ gets closer to the slope of the tangent at P. The *gradient* of the tangent at the point P is the limit:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \tag{50}$$

If such a limit is finite, then we say that the function f is *differentiable* at x with derivative equal to the limit which is denoted by dy/dx or f'. Thus

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \tag{51}$$

The function f' is called the derivative of the function f. In the example above the speed of the object $v(t)$ is the derivative of the distance s and is equal to $-9.8t$.

Not all functions are differentiable everywhere in their domains. For example, the function $\sqrt{x}$ is not differentiable at $x = 0$ since $\delta y/\delta x$ does not tend to a finite limit as δx tends to 0. This corresponds to the graph of the function $y = \sqrt{x}$ having a vertical tangent at (0, 0).

There are many examples of the use of differentiation and derivatives in physics.

- The instantaneous electric current is defined as

$$I = \frac{dQ}{dt} \tag{52}$$

  where Q is the charge passing through some point, and t is the time.

- The electric field intensity (strength) at any point can be expressed in terms of the potential gradient dV/dx – the rate of change of potential with distance at this point

$$E = \frac{dV}{dx}. \tag{53}$$

- The Newton's second law is

$$F = m\frac{dv}{dt} \tag{54}$$

  where F is the force acting on a body, m is the body's mass and dv/dt is the instantaneous change in velocity or acceleration.

- The bulk modulus of elasticity of a material is defined as

$$K = \frac{dp}{dV} \tag{55}$$

  where dp is the change in pressure and dV is the change in volume.

Sometimes the *time* derivatives of the functions are denoted as $\dot{y}$ or $\dot{f}(t)$.

7 Rules of differentiation

7.1 Differentiation of a constant

If $y = constant$, then $\delta y = 0$ and $dy/dx = 0$. The inverse statement is also true. Thus: **the derivative of a constant function is zero. If the derivative of a function is zero, then the function is a constant**.

7.2 Differentiation of a function $y = x$

If $y = x$, then $\delta y = \delta x$ and $dy/dx = 1$. Thus

$$\frac{d}{dx}(x) = 1 \tag{56}$$

7.3 The sum rule

Let u and v be differentiable functions of x and let a and b be constants. Then the function $au+bv$ is differentiable with derivative $a\frac{du}{dx}+b\frac{dv}{dx}$. So

$$(au+bv)' = au' + bv' \tag{57}$$

In particular, $(u+v)' = u' + v'$, $(u-v)' = u' - v'$, $(bu)' = bu'$.

7.4 The product rule

Consider the product of two functions $y(x) = u(x)v(x)$.

$$\delta y = u(x+\delta x)v(x+\delta x) - u(x)v(x) = (u+\delta u)(v+\delta v) - uv = u\delta v + v\delta u + \delta u \delta v \tag{58}$$

$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u \delta v}{\delta x} \tag{59}$$

Taking $\delta x \to 0$ we get $\delta u/\delta x \to du/dx$, $\delta v/\delta x \to dv/dx$ and $\delta u\delta v/\delta x \to 0$ (you can consider this as $\delta v du/dx \to 0$ because $\delta v \to 0$, or $\delta u dv/dx \to 0$ because $\delta u \to 0$ – if $\delta x \to 0$, then $\delta u \to 0$ and $\delta v \to 0$). So

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \tag{60}$$

or

$$(uv)' = uv' + u'v \tag{61}$$

7.5 The quotient rule

Similarly the quotient rule

$$\frac{d}{dx}\frac{u}{v} = \frac{(du/dx)v - u(dv/dx)}{v^2} \tag{62}$$

or

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \tag{63}$$

If $u = 1$, then

$$\left(\frac{1}{v}\right)' = \frac{-v'}{v^2} \tag{64}$$

7.6 Differentiation of x^n

From the product rule:

$$\frac{d}{dx}(x^2) = x + x = 2x \tag{65}$$

In general, for any real a,

$$\frac{d}{dx}(x^a) = ax^{a-1} \tag{66}$$

7.7 Differentiation of a composite function: the chain rule

Suppose there is a function $y = f(u)$ and $u = g(x)$, so $y = f(g(x))$ – a composite function considered earlier. Let x change by an amount δx with the corresponding changes in u and y being δu and δy. Then, in general,

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u}\frac{\delta u}{\delta x} \tag{67}$$

As $\delta x \to 0$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{or} \quad (f(g(x)))' = f'(u)g'(x) \tag{68}$$

This is called the *chain rule.*

Example:

Differentiate $y = (5x^3 - 7)^4$.

Let us assume that $u = 5x^3 - 7$. Then use the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3 \times 15x^2 = 60x^2(5x^3 - 7)^3 \tag{69}$$

7.8 Extended chain rule

Suppose there is a function $y = f(v)$, where $v = g(u)$ and $u = h(x)$, so that $y = f(g(h(x)))$ Then the *chain rule* can be extended to include more complicated composite functions (a function of a function of a function...). The *extended chain rule* is then:

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} \tag{70}$$

7.9 Differentiation of trigonometric functions

We need additional trigonometric inequalities to learn how to differentiate trigonometric functions. Figure 9 shows a circle with a radius $r = 1$ and angle x is measured in radians: $0 < x < \pi/2$.

It is obvious that

area $\triangle BOA <$ area sector $BOA <$ area $\triangle COA$, i.e.

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x \tag{71}$$

(Recall that the area of the triangle BOA can be calculated as $\frac{1}{2}BO \times OA \times \sin x$, the area of the sector BOA is $\frac{1}{2}BO^2 \times x$ and the area of the triangle COA is $\frac{1}{2}CA \times OA = \frac{1}{2}OA^2 \tan x$, where $OA = BO = 1$).

Multiplying all parts by $2/\sin x$ gives:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \tag{72}$$

Now taking reciprocals:

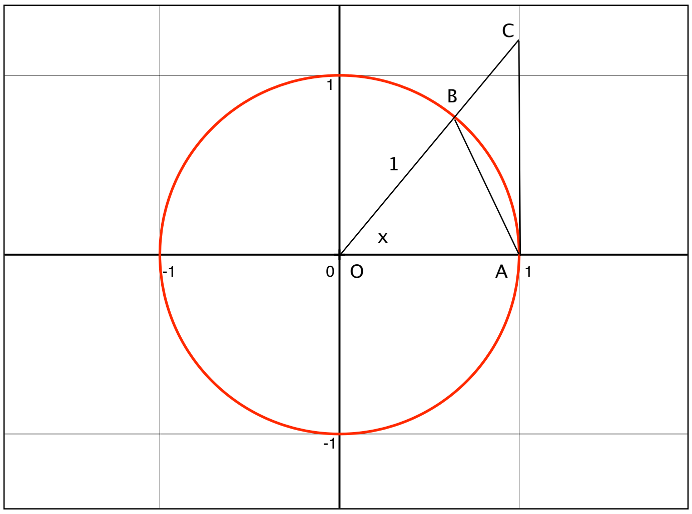

Figure 9: Illustration of the limit $\sin x/x$

$$1 > \frac{\sin x}{x} > \cos x \tag{73}$$

Tending $x \to 0$ and taking into account that $\cos x \to 1$ as $x \to 0$, we deduce a limit:

$$\frac{\sin x}{x} \to 1 \quad \text{as} \quad x \to 0 \tag{74}$$

Now consider the function $y = \sin x$ and find its derivative:

$$\begin{aligned} \frac{\delta y}{\delta x} &= \frac{\sin(x+\delta x) - \sin x}{\delta x} \\ &= \frac{2\cos(\frac{1}{2}((x+\delta x)+x)) \times \sin(\frac{1}{2}((x+\delta x)-x))}{\delta x} \\ &= \cos(x + \frac{1}{2}\delta x) \times \frac{\sin(\frac{1}{2}\delta x)}{(\frac{1}{2}\delta x)} \\ &\to \cos x \quad \text{as} \quad \delta x \to 0 \end{aligned} \tag{75}$$

Thus

$$\frac{d}{dx}(\sin x) = \cos x \tag{76}$$

In a similar way the derivatives of other trigonometric functions can be derived. However, there is more elegant way of finding them.

Exercises:

1. Using relations between trigonometric functions, derivative of $\sin x$ and the chain rule find the derivatives of $\cos x$ and $\tan x$.

2. Using the extended chain rule find the derivative of $\cos\left((3x^2+1)^3\right)$.

7.10 Differentiation of the inverse functions

If y is a differentiable function of x and $y = f(x)$ has an inverse function $x = u(y)$ then as x changes by δx and y changes by δy we have

$$\frac{dx}{dy} = \lim_{\delta y \to 0} \frac{\delta x}{\delta y} = \lim_{\delta x \to 0} \frac{1}{\frac{\delta y}{\delta x}} = \frac{1}{\frac{dy}{dx}} \tag{77}$$

Thus the inverse function rule is:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \tag{78}$$

7.11 Differentiation of inverse trigonometric functions $\arcsin x$

Let $y = \arcsin x = \sin^{-1} x$. Then $x = \sin y$ and for $-\pi/2 \leq y \leq \pi/2$

$$\frac{dx}{dy} = \cos y \quad \text{so that} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - (\sin y)^2}} = \frac{1}{\sqrt{1 - x^2}} \tag{79}$$

Exercises:

1. Find the derivative of $\arccos x$, $\arccos ax$, $\arctan x$.

7.12 Differentiation of an exponential function

Remember that we defined e as:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = 1 + 1 + \frac{1}{2} + \frac{1}{6} + ... \tag{80}$$

or the function e^x as:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... \tag{81}$$

Applying the sum rule of differentiation to the equation above

$$\begin{aligned} \frac{d}{dx}(e^x) &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + ... + \frac{nx^{n-1}}{n!} + ... \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + ... = e^x \end{aligned} \tag{82}$$

So the function e^x is equal to its own derivative. In fact we can reverse the definition and the consequence and define the number e in such a way that the function e^x is equal to its own derivative.

Thus

$$\frac{d}{dx}(e^x) = e^x \tag{83}$$

Examples:

a) Assuming a radioactive isotope decays according to the exponential law $I(t) = I_0 e^{-t/t_0}$, where I is the number of radioactive atoms, find the rate of decay of this isotope as a function of time.

The rate of decay can be treated similarly to the velocity of an object:

$$R = \lim_{\delta t \to 0} \frac{\delta I}{\delta t} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t} = \frac{dI}{dt} \tag{84}$$

Thus

$$\frac{dI}{dt} = -\frac{I_0}{t_0} e^{-t/t_0} = -\frac{I(t)}{t_0} \tag{85}$$

So the rate of decay is proportional to the number of radioactive atoms present and the coefficient (decay constant) is equal to the reciprocal of the parameter in the exponential t_0, which is known as the lifetime of the isotope (connected to the half-life). The minus sign indicates that the number of radioactive atoms decreases with time. (In fact the exponential law of radioactive decay was established by Rutherford and Soddy in 1903 by observing that the rate of decay is proportional to the amount of radioactive material present in their experiments).

b) The Maxwell-Boltzmann distribution describes the distribution (spectrum) of energies (E) of molecules in a gas and its variation as a function of temperature (T) (k is the Boltzmann constant):

$$f(E,T) = 2\sqrt{\frac{E}{\pi (kT)^3}} \times e^{-\frac{E}{kT}} \tag{86}$$

Find the derivative to this spectrum (for fixed T).

Using the product rule and the chain rule we have $f(E, T = const) = u(E)v(E)$:

$$\begin{aligned} f'(E, T = const) &= \frac{\frac{1}{\pi (kT)^3}}{\sqrt{\frac{E}{\pi (kT)^3}}} \times e^{-\frac{E}{kT}} \\ &+ 2\sqrt{\frac{E}{\pi (kT)^3}} \left(-\frac{1}{kT}\right) \times e^{-\frac{E}{kT}} \\ &= e^{-\frac{E}{kT}} \times \left(\frac{1}{\sqrt{E\pi (kT)^3}} - \frac{2}{kT}\sqrt{\frac{E}{\pi (kT)^3}} \right) \end{aligned} \tag{87}$$

Let us find the most probable energy for this distribution – the energy at the peak of the spectrum (see Figure 10). This is one of the application of differentiation.

The slope or the gradient of the tangent to the graph at maximum (or minimum) is equal to 0 (the tangent is parallel to the x-axis), so the derivative then is also equal to 0. Thus

$$f'(E, T = const) = e^{-\frac{E}{kT}} \times \left(\frac{1}{\sqrt{E\pi (kT)^3}} - \frac{2}{kT}\sqrt{\frac{E}{\pi (kT)^3}} \right) = 0 \tag{88}$$

Simplifying this equation, we obtain $E_{peak} = kT/2$. We can also say that the function $f(x)$ is increasing if $f' > 0$ (positive slope or gradient of the tangent to the graph), and decreasing if $f' < 0$.

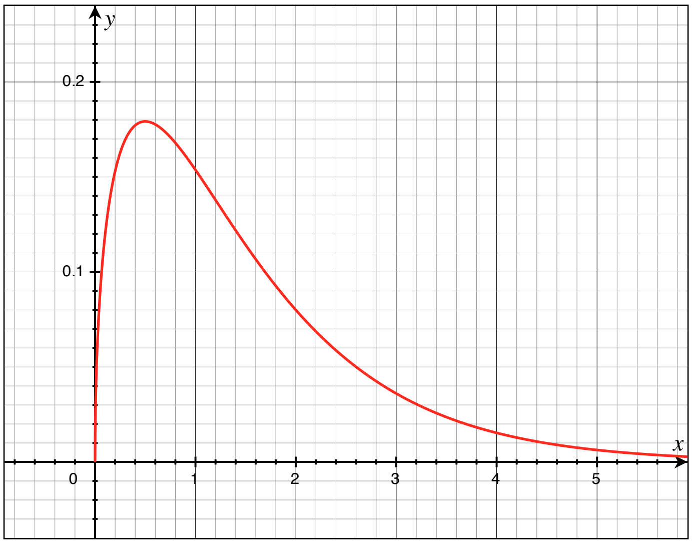

Figure 10: Example of the Maxwell-Boltzmann distribution

Exercises:

1. Find the derivative $\frac{d}{dT}\left(2\sqrt{\frac{E}{\pi(kT)^3}} \times e^{-\frac{E}{kT}}\right)$ assuming $E = const$.

2. Find the derivative of the Planck spectrum:

$\frac{d}{df}\left(\frac{8\pi h}{c^3}\frac{f^3}{e^{\frac{hf}{kT}}-1}\right)$,

where f is the frequency of radiation, h is the Planck constant and c is the speed of light in vacuum, assuming $T = const$. This spectrum is very well known in physics. It describes energy (frequency) distribution of bosons – particles with integer spin obeying Bose-Einstein statistics. The Planck formula gives the spectrum of black-body radiation, an example of which is the cosmic microwave background radiation. (You will learn all this later on studying Statistical Physics and Cosmology – there is no need to remember everything now but you should know how to find the derivative of this).

7.13 Differentiation of a logarithmic function

If $y = \ln x$, then $x = e^y$ and using inverse function rule

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x} \tag{89}$$

Thus

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \tag{90}$$

7.14 Logarithmic differentiation

To differentiate certain functions may require taking the logarithm first. For example, the function in the form $y = (f(x))^{g(x)}$ can be differentiated in the following way. Let us take the (natural) logarithm first,

$$\ln y = \ln\left((f(x))^{g(x)}\right) = g(x)\ln(f(x)) \tag{91}$$

Then we differentiate this equation using the chain rule, obtaining for the left-hand side,

$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx} \tag{92}$$

and for the right-hand side,

$$\frac{d}{dx}\left(g(x)\ln(f(x))\right) = g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)} \tag{93}$$

From this we now derive

$$\frac{dy}{dx} = (f(x))^{g(x)}\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right) \tag{94}$$

Example:

Find the derivative of $y = a^x$ $(a > 0)$.

First we take a logarithm $\ln y = x\ln a$, then differentiate this equation $\frac{1}{y}y' = \ln a$, from where we can derive $y' = a^x \ln a$. The same answer can be obtained by putting $f(x) = a$ and $g(x) = x$ into Eq. (94). So we found that

$$\frac{d}{dx}(a^x) = a^x \ln a \tag{95}$$

Exercise:

Find the derivative of x^x.

Logarithmic differentiation can also be applied to a product of several functions instead of the product rule (if you prefer the product rule then it should be applied several times in this case). If there is a function $y = u(x)v(x)w(x)$, then by taking a logarithm we get $\ln y = \ln u + \ln v + \ln w$, and differentiating both sides of the equation we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} \tag{96}$$

Finally, recalling that $y = uvw$ we get

$$\frac{dy}{dx} = uvw\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right) \tag{97}$$

7.15 Parametric differentiation

If x and y are functions of a parameter, or supplementary variable, t, $x = f(t)$, $y = g(t)$, then we can present this as a three-dimensional graph, where t is an argument and both x and y are functions. As there is only one argument and two functions, the graph will be the curve in space. (This is different from the case of a function $z = f(x, y)$ which is a surface in space). We can also draw the projection of this graph on (x, y) plane and find a function $y = u(x)$. Sometimes, however, it may be difficult to find an analytical expression for this function whereas we may still need to differentiate it. So we can do this by differentiating the two functions with respect to the parameter. If t changes from t to $t+\delta t$, then x changes from x to $x+\delta x$ and y changes from y to $y+\delta y$. Obviously $\frac{\delta y}{\delta x}=\frac{\delta y}{\delta t}/\frac{\delta x}{\delta t}$, since δt cancels on the right-hand side. If $\delta t\to 0$, then we can write:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{98}$$

Examples:

a) Find $\frac{dy}{dx}$ if $y=\ln t-t^3$ and $x=\cos t+e^t$.

Use parametric differentiation:

$\frac{dy}{dt}=\frac{1}{t}-3t^2;\qquad \frac{dx}{dt}=-\sin t+e^t$

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{1}{t}-3t^2}{-\sin t+e^t}=\frac{1-3t^3}{t(e^t-\sin t)}$

b) The curve is given in polar coordinates by $r=\sin\theta$. Find the tangent to the curve in cartesian coordinates.

The relation between the polar and cartesian coordinates are: $x=r\cos\theta$, $y=r\sin\theta$. So we can find x and y as functions of parameter - polar angle θ: $x=\sin\theta\cos\theta$, $y=\sin^2\theta$. Then differentiating the two functions we get: $\frac{dx}{d\theta}=\cos^2\theta-\sin^2\theta=\cos 2\theta$ and $\frac{dy}{d\theta}=2\sin\theta\cos\theta=\sin 2\theta$. So $\frac{dy}{dx}=\frac{dy}{d\theta}/\frac{dx}{d\theta}=\tan 2\theta$.

Another example related to the functions of a parameter (but not exactly to the parametric differentiation) is the motion of a car on a road. A car is moving on a road and the car's coordinates as a function of time are given by functions $x=u(t)$ and $y=v(t)$. Find the speed of a car as a function of time.

We know that the speed of the car along x-axis (in x direction) is given by $\frac{\delta x}{\delta t}$, whereas its speed along y-axis is $\frac{\delta y}{\delta t}$, where δx and δy are short distances travelled by a car along x and y axes over the time δt. It is obvious that the total distance is $\delta s\approx\sqrt{\delta x^2+\delta y^2}$ if the distances are short and we can neglect the non-linear motion of the car. So if $\delta t\to 0$, then $\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$.

7.16 Standard derivatives

Here is the list of standard derivatives which have to be learnt.

$\frac{d}{dx}(a)=0$ if a is a constant.

$\frac{d}{dx}\left(x^a\right)=ax^{a-1}$ for $a\neq 0$.

$\frac{d}{dx}\left(e^x\right)=e^x$.

$\frac{d}{dx}\left(a^x\right)=a^x\ln a$.

$\frac{d}{dx}\left(\ln x\right)=\frac{1}{x}$.

$\frac{d}{dx}\left(\log_a x\right)=\frac{1}{x\ln a}$.

$\frac{d}{dx}\left(\sin x\right)=\cos x$.

$\frac{d}{dx}\left(\cos x\right)=-\sin x$.

$\frac{d}{dx}(\tan x) = \sec^2 x$.

$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.

$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$.

$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

$\frac{d}{dx}(\sinh x) = \cosh x$.

$\frac{d}{dx}(\cosh x) = \sinh x$.

You should also remember how to use the rules of differentiation, such as the sum rule, the product rule, the chain rule, the logarithmic differentiation and the differentiation of parametric functions. Using these rules you should be able to calculate easily the derivatives such as:

$\frac{d}{dx}(\sin(x^2)) = 2x\cos(x^2)$.

$\frac{d}{dx}(\sec x) = \frac{\sin x}{\cos^2 x}$.

$\frac{d}{dx}(\arcsin \frac{x}{a}) = \frac{1}{\sqrt{a^2-x^2}}$.

See also Recommended textbook, Chapters 15-16, Pages 428-507, exercises on pages 442, 444, 446, 457, 459, 460, 465, 475, 479, 498.

8 Higher order derivatives and Taylor series

8.1 Higher order derivatives

The differentiation of a function f gives the first derivative $f' = \frac{dy}{dx}$. This may itself be differentiated to give the 2nd derivative of f denoted by $f'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. In general, the derivative of any order of a function $y = f(x)$ is denoted by $\frac{d^ny}{dx^n} = f''^{\ldots\prime}(x) = f^{(n)}(x)$.

An example of the second order derivative is the acceleration of the moving object. The first derivative describes the speed of the object v – the change of the position x (distance from the initial point): $v = dx/dt$. The second derivative is the change in the speed – acceleration: $a = dv/dt = d^2x/dt^2$.

Consider an example of an object falling down from the height of 20 meters (see the first lecture on differentiation). Its distance from the surface is described by the formula: $x = f(t) = 20 - 4.9t^2$ (we neglect here the air resistance). The speed of the object as a function of time has been found before and is $v = dx/dt = -9.8t$. The minus sign indicates that the object is moving towards decreasing the height x. Obviously the speed has the range of [-19.8,0] m/s because $x \geq 0$. We can differentiate v: $dv/dt = d^2x/dt^2 = -9.8$ m/s^2. The minus sign here indicates that the speed (negative) is decreasing, although the absolute value of speed in increasing.

On a graph of a function the second derivative corresponds to a change in the gradient or the slope of the tangent to this function. If the slope increases then the 2nd derivative is positive, if the slope decreases, then the 2nd derivative is negative, if the slope does not change, then the 2nd derivative is 0. For example, the slope of the straight line does not change and its 2nd derivative should be 0: $y = ax + b$; $dy/dx = a$; $d^2y/dx^2 = 0$.

When considering the maxima and minima of the functions, the 1st derivative equals to 0 at any extreme, whereas the sign of the 2nd derivative shows whether the extreme is the minimum or maximum. The 2nd derivative of the function is positive at minimum of the function (the slope goes from negative values through 0 to positive values, so increases), and is negative at maximum.

For parametric differentiation $x = f(t)$ and $y = g(t)$ the following rule applies for the 2nd and higher order derivatives:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right)/\frac{dx}{dt} \quad \text{etc.} \tag{99}$$

8.2 Taylor series

Taylor series or *Taylor expansion* describes the **polynomial approximation to a function** $f(x)$ **at any value of** x **in the** *interval of validity* for the series. Firstly we obtain approximations for small $x \approx 0$. Polynomial approximation at $x \approx 0$ are also called Maclaurin's series. Let us consider an example

$$f(x) = \frac{1}{1-x} \tag{100}$$

Since $f(0) = 1$, we can write

$$f(x) = \frac{1}{1-x} \approx 1 \tag{101}$$

as long as x is small enough. We can say that our approximation $y = 1$ equals the original function $f(x)$ at $x = 0$: $f(0) = y(0)$. So the first term in the series is 1 (see Figure 11). This approximation is quite poor, acceptable only if x is very close to 0. A better approximation can be given by the equation of the tangent line at $x = 0$. Since $f^{(1)} = 1/(1-x)^2$, the slope at $x = 0$ is $f^{(1)}(0) = 1$. If we require now that the derivative of the original function is equal to the derivative of its approximation at $x = 0$, namely: $f^{(1)}(0) = y^{(1)}(0)$, then we can find the coefficient for the second term of polynomial: $a_1 = 1$. The approximation now becomes

$$f(x) = \frac{1}{1-x} \approx 1 + x \tag{102}$$

when x is small enough. Note that the new term in x does not disturb the first condition $f(0) = y(0)$.

It turns out that this can be improved by taking the second derivative of the original function $f(x)$ and matching it to the derivative of the polynomial of the 2nd order (adding a term in x^2) at $x = 0$. The 2nd derivative is $f^{(2)}(x) = 2/(1-x)^3$, and $f^{(2)}(0) = 2$. We have now the 2nd order polynomial $y(x) = 1 + x + ax^2$ where the coefficient a should satisfy the requirement: $f^{(2)}(0) = y^{(2)}(0)$. This condition can be satisfied without disturbing two other terms previously found from conditions: $f(0) = y(0)$, $f^{(1)}(0) = y^{(1)}(0)$. This coefficient is equal to 1 (you can check this) and the approximation now becomes:

$$f(x) = \frac{1}{1-x} \approx 1 + x + x^2 \tag{103}$$

when x is small enough. This is a parabolic function also shown in Figure 11.

We can carry out this process for any function to any level of approximation but we need a general formula for any order derivatives of a polynomial $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ at $x = 0$. The derivatives are $P(0) = a_0$, $P^{(1)}(0) = a_1$, $P^{(2)}(0) = 2!a_2$ and in general $P^{(n)}(0) = n!a_n$.

If we want to approximate to a general function $f(x)$ at $x \approx 0$ by means of a polynomial $P(x) = a_0+a_1x+a_2x^2+...+a_nx^n$, then we first require that $f(0) = P(0)$, $f^{(1)}(0) = P^{(1)}(0)$, $f^{(2)}(0) = P^{(2)}(0)$

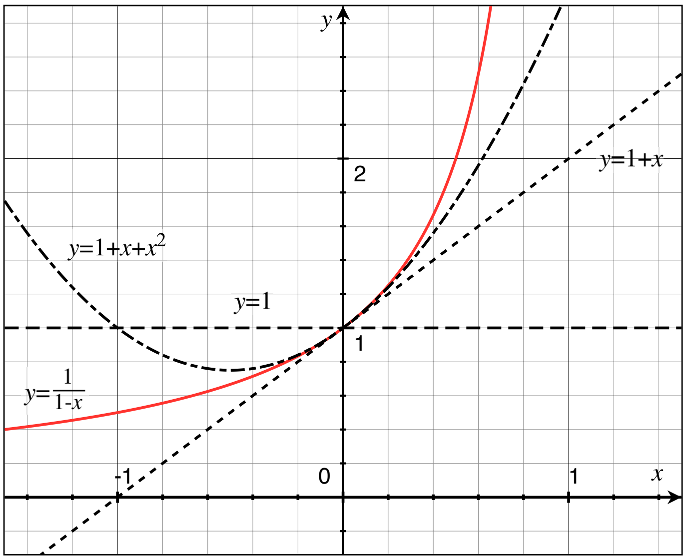

Figure 11: Illustration to the Taylor expansion

etc. According to the general rule for the derivatives of the polynomial (see above) the coefficients are given by $a_0 = P(0) = f(0)$, $a_1 = \frac{1}{1!}P^{(1)}(0) = \frac{1}{1!}f^{(1)}(0), a_2 = \frac{1}{2!}P^{(2)}(0) = \frac{1}{2!}f^{(2)}(0)$ etc. So the *Taylor polynomial approximation* of a degree n to a function $f(x)$ at $x \approx 0$ is given by

$$f(x) \approx f(0) + \frac{1}{1!}f^{(1)}(0)x + \frac{1}{2!}f^{(2)}(0)x^2 + ... + \frac{1}{n!}f^{(n)}(0)x^n \tag{104}$$

So far we considered the Taylor expansion of degree n at $x \approx 0$. This particular assumption is not necessary in many cases. Recall that introducing the number e we used a polynomial (Taylor) expansion of the function e^x at $x \approx 1$. In general, polynomial approximation can be used for any differentiable function and any x from the *interval of validity* which defines the region where the approximation converges to the exact value of $f(x)$. Any particular approximation can be used only within its interval of validity. The degree of a polynomial is not restricted and in principle can be infinite. In this case we have an *infinite series* for which the sum of the series will be *equal* to the original function – approximation becomes an equality. Again we used this already when we wrote an approximation for the function e^x. Now we can show that this is a Taylor series.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ... \tag{105}$$

Other important examples of Taylor series:

1. Geometric series (valid for $-1 < x < 1$):

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + ... \tag{106}$$

where n is an integer number: $n \geq 0$.

2. Trigonometric series (valid for all x in radians):

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + ... \tag{107}$$

where n is an integer number: $n \geq 1$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + \frac{(-1)^n x^{2n}}{(2n)!} + ... \tag{108}$$

where n is an integer number: $n \geq 0$.

3. Logarithmic series (valid for $-1 < x \leq 1$):

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \frac{(-1)^{n-1}x^n}{n} + ... \tag{109}$$

where n is an integer number: $n \geq 1$.

4. Binomial series (valid for $-1 < x < 1$ and any a):

$$\begin{aligned}(1+x)^a &= 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 \\ &+ \frac{a(a-1)(a-2)(a-3)}{4!}x^4 + ... + \frac{\prod_{k=1}^{n}(a-k+1)}{n!}x^n + ...\end{aligned} \tag{110}$$

where k and n are integer numbers: $k \geq 1$ and $n \geq 1$. (Note that the general formula for the n-term of the series is invalid for the first term $n = 0$).

If $a = -1$, then from the binomial series Eq. (110) we obtain geometric series Eq. (106). If $a = n$ where n is a positive integer, then the series terminates at the term with x^n. Then we have the *binomial theorem* valid for all x:

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \\ &+ \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + ... + \frac{n!}{k!(n-k)!}x^k + ... + x^n\end{aligned} \tag{111}$$

8.3 Operations with Taylor series

New Taylor series can be obtained from the standard ones listed above.

1. Find Taylor expansion about $x = 0$ for the function $(2-x)^{1/2}$.

$$(2-x)^{1/2} = \left(2(1 - \frac{1}{2}x)\right)^{1/2} = 2^{1/2}\left(1 + (-\frac{1}{2}x)\right)^{1/2} \tag{112}$$

We can use the known binomial expansion given by Eq. (110) with $a = 1/2$ and with $-\frac{1}{2}x$ in place of x. The expansion will be valid for $-1 < -\frac{1}{2}x < 1$, that is for $-2 < x < 2$.

$$\begin{aligned}(2-x)^{1/2} &= 2^{1/2}\left(1 + (-\frac{1}{2}x)\right)^{1/2} \\ &= 2^{1/2}\left(1 + \frac{1}{2}(-\frac{1}{2}x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-\frac{1}{2}x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(-\frac{1}{2}x)^3 + ...\right) \\ &= 2^{1/2}\left(1 - \frac{1}{4}x - \frac{1}{32}x^2 - \frac{1}{128}x^3 + ...\right)\end{aligned} \tag{113}$$

2. Find Taylor series about $x = 0$ for the function $\frac{\sin x}{x}$.

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{(-1)^{n-1} x^{2n-2}}{(2n-1)!} + ... \tag{114}$$

8.4 Taylor series about points other than 0

In many cases it is necessary to find an approximation to a function at a point different from $x = 0$. For many functions we can still use the approximations described above. But what if the point x is beyond the interval of validity of the approximation? Let us consider again an example

$$f(x) = \frac{1}{1-x} \approx 1 + x + x^2 \tag{115}$$

This certainly works for $x \approx 0$. But suppose we need to find the approximation at $x \approx 2$. Then the series would give an infinite sum which does not converge to the original function. So $x = 2$ is out of the interval of validity of this approximation. The way around this problem is to use the polynomial consisting of powers of $x - 2$ instead of x. In this case $x - 2$ is small. In general, the Taylor series about a point $x = a$ is

$$f(x) \approx f(a) + \frac{1}{1!} f^{(1)}(a)(x-a) + \frac{1}{2!} f^{(2)}(a)(x-a)^2 + ... + \frac{1}{n!} f^{(n)}(a)(x-a)^n \tag{116}$$

The interval of validity depends on the function $f(x)$.

8.5 Approximations for large values of x

When x is large, $1/x$ is small. This can be used sometimes to obtain approximations valid for large values of x. For example, let us find approximation to the function $(1+1/x)^{1/2}$ for $x \to \infty$. We can make a substitution $u = 1/x$ (u is now small) and use the result for binomial series with $a = 1/2$:

$$(1+u)^{\frac{1}{2}} = 1 + \frac{1}{2}u + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} u^2 + ... = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + ... \tag{117}$$

Converting back to x gives:

$$\left(1 + \frac{1}{x}\right)^{\frac{1}{2}} = 1 + \frac{1}{2x} - \frac{1}{8x^2} + ... \tag{118}$$

8.6 Indeterminate values; l'Hôpital rule

Consider the function $f(x) = \frac{\sin x}{x}$. The function specifies a value for y for all values of x except $x = 0$ for which the function gives $y = 0/0$ - indeterminate value. However, the value for $y(0)$ can still be determined if we use a limit and Taylor expansion:

$$y(0) = \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x - \frac{1}{3!}x^3 + ...}{x} = \lim_{x \to 0} (1 - \frac{1}{3!}x^2 + ...) = 1 \tag{119}$$

The procedure described above required finding the first and higher order derivatives of the function in numerator (Taylor series is a polynomial that includes derivatives as coefficients). Sometimes you have to write down Taylor series for both numerator and denominator. Since Taylor expansion gives you polynomials in numerator and denominator, a simple rule exists to calculate the ratio of the two functions at a point where the value is indeterminate (by direct substitution). This is called *l'Hôpital rule.* **If the ratio of the two functions $f(x)/g(x)$ is indeterminate at $x = a$ ($f(a) = g(a) = 0$), then the limiting value of $f(x)/g(x)$ at $x \to a$ is given by the ratio of the derivatives of numerator and denominator (provided that the derivatives are not both zeros themselves):**

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \tag{120}$$

Before applying the rule, you have to check that the ratio $f(a)/g(a) = 0/0$.

If $f'(a)/g'(a) = 0/0$ as well, then the next order derivatives should be found and so on until either $f^{(n)}(a) \neq 0$ or $g^{(n)}(a) \neq 0$.

8.7 Application of Taylor series

There are many applications of Taylor series in Math, Physics, Engineering and other scientific disciplines. We will consider only two examples here.

1. Integration.

Let us consider an integral: $\int \frac{\sin x}{x} dx$. It is not easy to calculate it unless you use the Taylor expansion for $\sin x$.

$$\begin{aligned} \int \frac{\sin x}{x} dx &= \int \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...}{x} dx = \int \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ...\right) dx \\ &= x - \frac{x^3}{3 \times 3!} + \frac{x^5}{5 \times 5!} - ... + c \end{aligned} \tag{121}$$

2. The equivalence of mass and energy stated by Einstein gives the relation: $E = mc^2$, where E is the total (relativistic) energy and m is the relativistic mass. The energy E_0 associated with the rest mass m_0 is given by $E_0 = m_0 c^2$ and the relativistic mass is related to the rest mass by the equation: $m = m_0 \gamma$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. Show that if the velocity v of an object is much smaller than the speed of light in vacuum c, the kinetic energy of the object is given by the well known relation: $E_{kin} = m_0 v^2/2$.

The kinetic energy is (by definition) the total energy minus the rest energy: $E_{kin} = E - E_0 = m_0 c^2 \gamma - m_0 c^2 = m_0 c^2 \left((1 - v^2/c^2)^{-1/2} - 1\right)$. We can use Taylor series for the function $f(x) = (1 - x)^{-1/2}$ about 0 making a substitution $x = \beta^2 = v^2/c^2$.

$$\left[(1-x)^{-\frac{1}{2}}\right]_{x\approx 0} \approx \left[(1-x)^{-\frac{1}{2}}\right]_{x=0} + \left[-\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1)\right]_{x=0} x + ... = 1 + \frac{1}{2}x + ... \tag{122}$$

Substituting $x = v^2/c^2$ back into the expression for E_{kin} we get:

$$\begin{aligned} E_{kin} &= E - E_0 = m_0 c^2 \gamma - m_0 c^2 = m_0 c^2 \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right) \\ &\approx m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) = \frac{m_0 v^2}{2} \end{aligned} \tag{123}$$

See also Recommended textbook, Chapter 19, Pages 649-456, exercises on pages 652, 654, 655-656.